MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 10.5: The Ratio Test and a Summary of Convergence Tests

This section focuses on an important convergence test called the ratio test. This test will be particularly useful when discussing Power Series in the next section. We also provide a recap of all the convergence/divergence tests.

The Ratio Test

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \neq 0$ for each n. If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely. If the limit above is $r > 1$ (or infinite), the series diverges. If the limit above is $r = 1$, the test is inconclusive.

This test is particularly useful with terms involving $n!$ or exponential functions such as 2^n. The ratio test works because it compares the series to a geometric series. If the limit exists, then the series *eventually* looks geometric with ratio r. The notation $n!$, called "n **factorial**," is defined as

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1\,.$$

So for example, $3! = 6$, $4! = 24$ and $10! = 3{,}628{,}800$. The quantity $n!$ grows very, very fast. Consequently, $1/n!$ goes to zero very, very fast. One useful formula when dealing with factorials is

$$\boxed{(n+1)! = (n+1) \cdot n!}$$

For instance, $5! = 5 \cdot 4! = 5 \cdot 24 = 120$.

Example 1: Consider the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$. Note that the numerator is an exponential function, which grows very fast. But the $n!$ in the denominator goes to zero faster. Applying the ratio test, we compute

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2^n \cdot 2}{(n+1) \cdot n!} \cdot \frac{n!}{2^n} = \frac{2}{n+1}\,.$$

Thus, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2}{n+1} = 0$, so by the ratio test, the series converges.

Exercise 1: Use the ratio test to determine whether the given series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{(1.2)^n}$

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{1000^n}$

Summary of Convergence/Divergence Tests

1. **The nth Term Divergence Test:** If $\lim_{n\to\infty} a_n \neq 0$, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive. Try another test.

2. **Geometric Series:** A geometric series with first term a and ratio r satisfying $|r| < 1$ converges to the sum
$$\sum_{n=0}^{\infty} ar^n \;=\; a + ar + ar^2 + ar^3 + \cdots \;=\; \frac{a}{1-r}.$$
If $|r| \geq 1$, the series diverges.

3. **The p-series Test:** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

4. **The Integral Test:** Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n > 0$ for each n. Let $f : [1, \infty) \to \mathbb{R}$ be the function obtained by replacing the n in the formula for a_n with the variable x. Suppose that $f(x)$ is a positive, decreasing, and continuous function. Then
$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{if and only if} \quad \int_1^{\infty} f(x)\,dx \text{ converges.}$$

5. **The Comparison Test:** Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences satisfying $0 \leq a_n \leq b_n$ for each n.
If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. Equivalently, if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

6. **The Limit Comparison Test:** Suppose that $\{a_n\}$ and $\{b_n\}$ are two positive sequences and that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ exists with $L > 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

7. **The Ratio Test:** Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \neq 0$ for each n. If
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \;=\; r \;<\; 1,$$
then the series converges absolutely. If the limit above is $r > 1$ (or infinite), the series diverges. If the limit above is $r = 1$, the test is inconclusive.

8. **The Absolute Convergence Test:** If the infinite series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

9. **The Alternating Series Test:** Suppose that $\{a_n\}$ is a decreasing sequence of positive numbers that converges to 0. Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + - \cdots$ converges.

Exercises: Determine whether the given series converges or diverges using an appropriate test. Be sure to justify your test works (e.g., the assumptions are satisfied).

2. $\displaystyle\sum_{n=3}^{\infty} \frac{n^2 - 5}{n^5 + n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^2}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$

6. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$