MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 10.1: Sequences

Chapter 10 concerns the subject of infinite series, where we sum a list of numbers that is **infinitely** long. It is a fascinating topic in calculus with surprising results and famous formulas. In order to make sense of an infinite sum, we first need to learn about sequences and what it means for a sequence to converge or diverge.

Example 1: The Fibonacci Sequence

Consider the list of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots .$$

This is a famous, infinite list of numbers known as the **Fibonacci Sequence**, named after the mathematician Leonardo Pisano (1175-1250). Fibonacci, which loosely translates as "son of Bonacci," was a pseudonym for Pisano. Before Fibonacci, Indian scholars such as Gopala (before 1135) and Hemachandra (1089–1172) discussed this sequence in their analysis of Indian rhythmic patterns. Surprisingly, the Fibonacci numbers frequently arise in nature. For instance, the number of petals in many flowers are Fibonacci numbers: 3-leaf clover, buttercups (5), black-eyed Susan (13), chicory and Shasta daisy (21) (see below).



Figure 1: Some Fibonacci flowers: Columbine (left, 5 petals); Black-eyed Susan (center, 13 petals); Shasta Daisy (right, 21 petals);

In general, a **sequence** is an ordered, infinite list of numbers. We use the notation a_n to represent the nth **term** (number) in the sequence. So for the Fibonacci sequence, we have $a_1 = 1$ because the first term in the list is 1. Similarly, $a_4 = 3$ and $a_{10} = 55$. The subscript n is called the **index** since it identifies a specific term in the sequence. Notice that the Fibonacci sequence satisfies the equation

$$a_{n+2} \;=\; a_{n+1} + a_n,$$

since the next term in the sequence is obtained by summing the previous two terms. This is an example of a **recursive sequence**, where a formula is specified to define the next numbers in a sequence using the previous values.

Exercise 1: Let a_n be the sequence defined by the relation $a_{n+1} = a_n^2 - 1$ and $a_1 = 2$. What are the next four terms in the sequence?

Typically we define a sequence using a formula that depends on the index n. For example, the sequence $a_n = n$ defines the sequence of counting numbers:

$$a_n = n \qquad \Longrightarrow \qquad 1, 2, 3, 4, 5, 6, \ldots$$

The sequence $a_n = 2n - 1$ gives the sequence of odd numbers:

$$a_n = 2n - 1 \qquad \Longrightarrow \qquad 1, 3, 5, 7, 9, 11, \ldots$$

The ... indicates that the pattern continues forever.

Example 2: A Convergent Sequence

Consider the sequence defined by $a_n = \dfrac{n}{n+1}$:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$$

If we convert these terms into decimals we obtain

$$0.5,\ 0.66\overline{6},\ 0.75,\ 0.8,\ 0.833\overline{3}, 0.8571,\ \ldots$$

The numbers appear to be heading toward 1. This can be confirmed mathematically by evaluating the limit of the sequence as $n \to \infty$. For instance, if $n = 99$, we have $a_{99} = 99/100 = 0.99$ and if $n = 999{,}999$, we have $a_{999{,}999} = 999{,}999/1{,}000{,}000 = 0.999999$. Since

$$\lim_{n\to\infty} a_n \;=\; \lim_{n\to\infty} \frac{n}{n+1} \;=\; 1\,,$$

we say the sequence **converges** to 1. Note that the limit can be computed with L'Hôpital's Rule.

Definition: Given a sequence a_n, we say the sequence **converges to the limit** L if $\lim\limits_{n\to\infty} a_n = L$. If the limit is ∞ or if the limit does not exist, we say the sequence **diverges**.

Exercise 2: Consider the two sequences defined by $a_n = (\frac{2}{3})^n$ and $b_n = (\frac{3}{2})^n$. Write out the first four terms of each sequence and then determine whether each sequence converges or diverges. If it converges, state the limit of the sequence.

Geometric Sequence:

Each of the sequences in the previous exercise are geometric sequences. A **geometric sequence** is one of the form $a_n = c\,r^n$ for some nonzero constants c and r. The number r is called the **ratio**.

Key Fact: The geometric sequence $a_n = c\,r^n$ converges to 0 if $|r| < 1$. If $r = 1$, the sequence converges to c. In all other case, the sequence diverges.

Exercises:

3. Find a formula for each of the following sequences (e.g., $a_n = 2n^2 - 3$).

   (a) 2, 4, 6, 8, 10, 12, ...

   (b) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

   (c) $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \frac{1}{26}, \ldots$

   (d) $-1, 1, -1, 1, -1, 1, \ldots$

4. For each of the following sequences, write out the first four terms of the sequence. Then determine whether the sequence converges or diverges. If it converges, state the limit of the sequence.

   (a) $a_n = n\cos(n\pi)$

   (b) $b_n = \left(\frac{3}{\pi}\right)^n$

   (c) $c_n = \left(\frac{\pi}{3}\right)^n$

   (d) $a_n = \dfrac{n}{\sqrt{4n^2 + 3}}$

   (e) $a_n = \dfrac{n}{2^n}$

   (f) $b_n = \tan^{-1}\left(\dfrac{3}{n^2} - 1\right)$

   (g) $s_n = \sin(n)$

   (h) $a_{n+1} = \sqrt{2a_n}\,, a_1 = 1$