MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 10.4: Absolute and Conditional Convergence

In this section we learn some useful tests for determining whether a series with both positive and negative terms converges or diverges. Recall that convergence tests such as the integral or comparison test require that the terms all be positive. Here we consider series where the terms may be either positive or negative, with particular attention to series where the terms alternate in sign.

Definition: Absolute and Conditional Convergence

An infinite series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges. On the other hand, the series **converges conditionally** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Note that $a_n \leq |a_n|$ always, so that considering the series of absolute values is considering a series with larger terms. Having a mixture of positive and negative terms in a series (e.g., alternating $+-+-+-\ldots$) is *useful* for convergence as it helps the sequence of partial sums converge. Think of the shopping cart metaphor; if you alternate between buying an item and receiving store credit, you have a much better chance of converging, assuming the terms are approaching zero. (The nth term divergence test still applies to series with terms of different signs.) Note also that if a series has only positive terms, or a finite number of negative terms, then absolute convergence is equivalent to regular convergence.

Example 1: The following are all examples of absolutely convergent series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \quad \sum_{n=1}^{\infty} \frac{\sin n}{n^3}.$$

Why? The first series, which converges because it is a p-series with $p = 2$, contains only positive terms so taking the absolute value of each term does not change the series. The second series becomes equal to the first one when taking the absolute value of each term because $|(-1)^n| = 1$. Thus this series is absolutely convergent because the first series converges. Using the fact that $|\sin n| \leq 1$ for any n, we have

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^3}\right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3},$$

which converges as a p-series with $p = 3$. Thus, $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^3}\right|$ converges by the comparison test.

Example 2: The standard example of a conditionally convergent series is the **Alternating Harmonic Series**, defined as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - \cdots$$

This series converges by the Alternating Series Test (see next page) and its sum is exactly $\ln 2$. However, taking the absolute value of each term gives the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

The Absolute Convergence Test

If the infinite series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

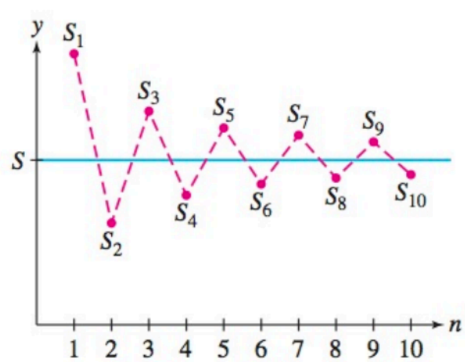
This is a useful test to apply when considering series that have some negative terms. Sometimes taking the absolute value of the terms in a series makes the series easier to understand and prove convergence. The converse of this test is false, as demonstrated by the alternating harmonic series. Just because $\sum_{n=1}^{\infty} a_n$ converges does **not** imply that $\sum_{n=1}^{\infty} |a_n|$ converges. In fact, any *conditionally convergent* series would violate the converse of this test.

Exercise 1: Use the absolute convergence test to show that $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges.

The Alternating Series Test

Suppose that $\{a_n\}$ is a decreasing sequence of positive numbers that converges to 0. Then the **alternating series** $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + - \cdots$ converges.

A decreasing sequence is one for which $a_{n+1} \leq a_n$, that is, the next term in the sequence is smaller or equal to the previous one. This is the best test to apply when considering the convergence of an alternating series. By plotting the sequence of partial sums, it is easy to believe the veracity of this test. The partial sums will oscillate back and forth because the signs alternate. However, since the terms are getting smaller and smaller in size, the partial sums will oscillate about, and approach, the limit of the series (see Figure 2). Note the difference between this test and the nth term divergence test.



DF **FIGURE 2** The partial sums of an alternating series zigzag above and below the limit. The odd partial sums decrease and the even partial sums increase.

Exercises:

2. Use the alternating series test to show that the Alternating Harmonic Series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - \cdots$$

converges.

3. Determine whether the given series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

(c) $\sum_{n=1}^{\infty} \frac{\sin(3n)}{n^{3/2}}$

(d) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

4. Determine whether the given series converges or diverges using an appropriate test.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+1}$

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{e^{n^3}}$