MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 10.3: Convergence of Series with Positive Terms

In this section we learn some tests for determining whether an infinite series converges or diverges. In general, it is not possible to find the explicit sum of a convergent series (exceptions are geometric and telescoping series); the main goal is to determine whether it converges or not. The tests in this section are only for series with *positive* terms.

The Integral Test

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n > 0$ for each n. Let $f : [1, \infty) \to \mathbb{R}$ be the function obtained by replacing the n in the formula for a_n with the variable x. Suppose that $f(x)$ is a positive, decreasing, and continuous function. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{if and only if} \quad \int_1^{\infty} f(x)\, dx \text{ converges.}$$

The idea behind the integral test is that a series with positive terms can be thought of as a Riemann sum with rectangles of base 1 and heights a_n. Thus, the improper integral should be a good approximation to the series. The integral and series converge or diverge together.

Example 1: Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We can apply the integral test by letting $f(x) = \frac{1}{x^2}$. Then f is a positive, decreasing, and continuous function for $x \geq 1$. It is decreasing because $f'(x) = -2x^{-1} < 0$ for $x \geq 1$. Since

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \int_1^b x^{-2}\, dx = \lim_{b\to\infty} -\frac{1}{x}\bigg|_1^b = \lim_{b\to\infty} -\frac{1}{b} + 1 = 1\,,$$

the improper integral converges. Therefore, by the integral test, the series also converges. Note that the series does **not** converge to the same value as the integral. In fact, the sum is actually $\pi^2/6$, a result that can be proved using Fourier Series.

Exercise 1: Use the integral test to determine whether the given series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

The p-series Test

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

A series of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a **p-series.** The p-series test follows directly from the integral test and the power rule. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges, while the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$ diverges. The border line case is $p = 1$, the all-important **Harmonic Series**. Of all the p-series, the Harmonic Series is the slowest divergent series (the sum goes to infinity very, very slowly—as slowly as $\ln x$ goes to infinity). The p-series test is particularly useful when applying the comparison test.

The Comparison Test

Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences satisfying $0 \leq a_n \leq b_n$ for each n.

If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Equivalently, if $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

This is pretty clear. Given two infinite series of positive terms, if the one with bigger terms converges, so does the smaller one. The contrapositive is that if the smaller one diverges, than so must the bigger one. It is worth pointing out that this theorem is still valid if the terms obey $0 \leq a_n \leq b_n$, for all $n \geq N$ for some natural number N. As long as the two series eventually obey the inequality, then the conclusion holds. Remember, it is the *tail* of the infinite series that matters not a finite number of terms at the start.

Example 2: Consider the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{n^3+1}$. We can apply the comparison test with the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2}$, which converges by the p-series test (or see Example 1). The constant 3 pulls out because the series is convergent. Let $a_n = \frac{3n}{n^3+1}$ and $b_n = \frac{3}{n^2}$. We must check that $a_n \leq b_n$ or

$$\frac{3n}{n^3+1} \leq \frac{3}{n^2}.$$

Multiplying through by $n^2(n^3+1)$, this is equivalent to checking that $3n^3 \leq 3n^3 + 3$, which is clearly true. Thus, since $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2}$ converges (the bigger series), so does $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{n^3+1}$, using the comparison test.

Exercise 2: Use the comparison test to determine whether the given series converges or diverges. *Hint:* What is the largest value that $\sin^2\theta$ can be?

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^4}$$

The Limit Comparison Test

Suppose that $\{a_n\}$ and $\{b_n\}$ are two positive sequences and that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ exists.

- If $L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
- If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.
- If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

The limit comparison test is useful when we are comparing a given series to a known convergent series, but the regular comparison test doesn't work (i.e., the inequality goes in the wrong direction). Taking the limit of the ratio of the terms of the two series is often an easier way to prove convergence.

Example 3: Consider the infinite series $\sum_{n=2}^{\infty} \frac{n^3}{n^5 - 2n^2 + 4}$. If we just focus on the higher order terms, this series looks like $\frac{n^3}{n^5} = \frac{1}{n^2}$. But the comparison test does not work because $\frac{n^3}{n^5 - 2n^2 + 4} > \frac{1}{n^2}$. However, if we let $a_n = \frac{1}{n^2}$ and $b_n = \frac{n^3}{n^5 - 2n^2 + 4}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n^5 - 2n^2 + 4}{n^3} = \lim_{n\to\infty} \frac{n^5 - 2n^2 + 4}{n^5} = \lim_{n\to\infty} 1 - \frac{2}{n^3} + \frac{4}{n^5} = 1 > 0.$$

Thus, by the limit comparison test, our series converges because $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges.

Exercises: Using an appropriate test for convergence, determine whether the given infinite series converges or diverges.

3. $\sum_{n=1}^{\infty} \frac{1}{n+4}$

4. $\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$

5. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$