# MATH 136-02, 136-03 Calculus 2, Fall 2018

## Section 10.2: Summing an Infinite Series

This section begins our study of infinite series. With a sequence, we are interested in whether an ordered list of numbers approaches a limit (converges). For an infinite series, we **sum** an ordered list of numbers and asks whether the sum converges. This is somewhat confusing at first glance so we will be sure to make a precise definition. Then we will learn how to sum two types of series: a geometric series and a telescoping series. We also discuss a famous series in mathematics called the Harmonic Series.

### Example 1: An Infinite Geometric Series

Consider the infinite sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

The $\cdots$ means to keep adding forever. However, it also means that the pattern continues. Notice that each term is precisely 1/2 the preceding term, so the terms are getting considerably smaller. Even though we are adding an infinite amount of numbers, the numbers are getting very small, very quickly. What does this sum equal? Is it finite or infinite?

This series is an example of a **geometric series** because the terms come from a geometric sequence. The common ratio between terms is $r = 1/2$. It turns out that this series converges to a sum of 1. The sum of the first two terms is $1/2 + 1/4 = 0.75$, and the sum of the first 10 terms is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{1024} \;=\; 0.9990234375\,.$$

The sum of the first 1000 terms is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{1000}} \;=\; 1 - \frac{1}{2^{1000}}\,.$$

In fact, if we add the first $n$ terms of the series, we obtain the sum

$$s_n \;=\; 1 - \frac{1}{2^n}\,.$$

No matter how many terms we add in the series, we are always slightly less than 1. Since $\lim_{n\to\infty} s_n = 1$, we say the infinite series converges to the sum 1.

### Sigma $\Sigma$ Notation

The previous series can be written more compactly as $\sum_{n=1}^{\infty} \frac{1}{2^n}$ or $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Here, $n$ is the **index** of the series and $\Sigma$ ("Sigma") indicates that we should plug in $n = 1, 2, 3, \ldots$ and sum together the results. It is important to remember that while a sequence is a *list* of numbers, a series is a *sum.*

In general we write

$$\sum_{n=1}^{\infty} a_n \;=\; a_1 + a_2 + a_3 + a_4 + \cdots$$

to represent an infinite series. Each **term** of the series is denoted by $a_n$. A series may start with a different value of the index other than $n = 1$.

**Definition: Convergence of an Infinite Series**

The infinite series $\sum_{n=1}^{\infty} a_n$ converges to the sum $S$ if the sequence of **partial sums** $s_n = a_1 + a_2 + \cdots + a_n$ converges to $S$. In this case, we say that

$$\sum_{n=1}^{\infty} a_n \; = \; S \quad \text{if and only if} \quad \lim_{n\to\infty} s_n \; = \; S.$$

If the sequence $s_n$ diverges, then the series is called **divergent**.

This definition is extremely important. Note how convergence of the infinite series depends on whether the sequence of partial sums converges or diverges. In the example above, the partial sum of the geometric series was $s_n = 1 - \frac{1}{2^n}$. This is a *finite* sum; it is the sum of the first $n$ terms of the series. Since $\lim_{n\to\infty} 1 - \dfrac{1}{2^n} = 1$, the geometric series converges to 1 and we are justified in claiming $\sum_{n=1}^{\infty} \dfrac{1}{2^n} = 1$.

One useful metaphor for infinite series (thanks to Prof. Levandosky) is an infinite shopping cart checking out at a mathematical supermarket. Each item you purchase is a term $a_n$. As the scanner records the price of each item, the subtotal (the sum of all items purchased thus far) is displayed on the screen for you and the cashier to see. The list of numbers being displayed on the screen is precisely the sequence of partial sums. Each item you purchase increases your total ($a_n > 0$ and $s_n > s_{n-1}$), while any coupons or store credit you have reduces your total ($a_n < 0$ and $s_n < s_{n-1}$). As you watch your infinite number of purchases go by, the question is whether the sequence of sub-totals converges or not. If it diverges, you either get in an argument with the cashier over the total price (because it never settles down to a particular value) or you go broke (because the sum is infinite).

Most theorems about infinite series can be deduced by applying the correct theorem about converging sequences to the sequence of partial sums. Note that it is the *tail* of the infinite series that matters when trying to determine convergence. Even if the first 2 million terms in the series are extremely large, the series can still converge if the remaining terms approach zero fast enough.

**Exercise 1:** Consider the infinite series $\sum_{n=1}^{\infty} (-1)^{n+1}$. Write out the first six terms in the series and then compute the first six partial sums: $s_1, s_2, s_3, s_4, s_5$, and $s_6$. Does the series converge or diverge? Explain.

Due to the simple structure of the terms in a geometric series (the next term is $r$ times the previous one), we can obtain explicit formulas for the partial sums and for the infinite sum itself. This is actually quite rare. For most convergent series, it is far easier to show convergence than it is to find an explicit formula for the sum of the series.

**Sum of a Geometric Series**

A geometric series with first term $a$ and ratio $r$ satisfying $|r| < 1$ converges to the sum

$$\sum_{n=0}^{\infty} ar^n \;=\; a + ar + ar^2 + ar^3 + \cdots \;=\; \frac{a}{1-r}.$$

If $|r| \geq 1$, the series diverges.

To derive the formula for the sum of a geometric series, we use a nice trick that highlights the underlying geometric principle. Beginning with the partial sum

$$s_n \;=\; a + ar + ar^2 + \cdots + ar^{n-1}, \tag{1}$$

multiply both sides by the common ratio $r$. Making sure to multiply every term on the right-hand side of equation (1) by $r$, we obtain

$$rs_n \;=\; ar + ar^2 + ar^3 + \cdots + ar^n. \tag{2}$$

Notice that nearly all of the terms on the right-hand side of equation (2) also appear on the right-hand side of equation (1). If we subtract equation (2) from equation (1), these terms cancel out (even the terms represented by the $\cdots$ are canceled). We then obtain

$$s_n - rs_n \;=\; a - ar^n, \tag{3}$$

an expression that is easily solved for $s_n$. Factoring out the $s_n$ on the left-hand side of equation (3) and dividing both sides by $1 - r$ gives an explicit formula for the sum of the first $n$ terms in a geometric series:

$$\boxed{s_n \;=\; \frac{a - ar^n}{1-r}.} \tag{4}$$

Since $\lim_{n\to\infty} r^n = 0$ whenever $|r| < 1$, we find that $\lim_{n\to\infty} s_n = \dfrac{a}{1-r}$, which verifies the formula for the sum of a convergent infinite geometric series.

**Exercise 2:** Find the sum of each of the following geometric series or state that the series diverges.

**(a)** $2 + \dfrac{4}{3} + \dfrac{8}{9} + \dfrac{16}{27} + \cdots$

**(b)** $8 - 2 + \dfrac{1}{2} - \dfrac{1}{8} + \dfrac{1}{32} - + \cdots$

**(c)** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-3}{5}\right)^n$

**(d)** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\pi}{3}\right)^n$

**Telescoping Series**

In addition to geometric series, there is another type of series for which it is possible to find the exact sum of the series. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots .$$

This series is not geometric because there is no common ratio between terms. However, notice that

$$\frac{1}{2} = 1 - \frac{1}{2}, \quad \frac{1}{6} = \frac{1}{2} - \frac{1}{3}, \quad \frac{1}{12} = \frac{1}{3} - \frac{1}{4}, \quad \text{and} \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

where the last expression can be obtained using partial fractions. It follows that the partial sum for the series can be simplified to

$$\begin{aligned} s_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}, \end{aligned}$$

since the second term in each parentheses is canceled out by the first term in the following parentheses. The series is called **telescoping** because the middle terms all cancel out and the partial sum collapses so nicely (think of an old-school telescope that sea captains once used). Since $\lim_{n\to\infty} s_n = 1$, the series converges to 1.

**Exercise 3:** Use partial fractions to find the sum of each telescoping series:

**(a)** $\displaystyle\sum_{n=0}^{\infty} \frac{3}{(n+1)(n+2)} = \frac{3}{2} + \frac{3}{6} + \frac{3}{12} + \frac{3}{20} + \cdots$

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots$

The following gives a test for the divergence of a series. If the terms being added to an infinite series do not approach zero, then the partial sums will continue to grow (or perhaps oscillate), and the series will diverge. From the shopping cart perspective, if we keep adding items whose price is not getting close to zero, then our total will either grow forever or perhaps oscillate continually.

**The $n$th Term Divergence Test**

If $\lim_{n\to\infty} a_n \neq 0$, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

**Exercise 4:** Use the $n$th term test to explain why the following series diverge:

**(a)** $\sum_{n=0}^{\infty} \frac{n}{3n-1}$

**(b)** $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+1}$

**(c)** $\cos 1 + \cos\frac{1}{2} + \cos\frac{1}{3} + \cos\frac{1}{4} + \cdots$

**Important:** The converse of the $n$th term test is false! Just because the terms in an infinite series go to zero, does *not* mean the series converges. The counterexample is the all-important Harmonic Series, which has terms converging to zero but still diverges to infinity (see below). Having the terms in an infinite series head to zero is *necessary* for convergence but not *sufficient*. You need to apply some other test to check for convergence.

**The Harmonic Series**

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots .$$

This is a very famous series in mathematics called the **Harmonic Series**. It is particularly surprising because the $n$th term is heading to zero, yet the series still diverges. To see this, consider the following calculation:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \cdots \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \cdots \\
&= 1 + \frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + 8\cdot\frac{1}{16} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots
\end{aligned}$$

Thus, the partial sums continue to grow by at least $\frac{1}{2}$ each time we group another $2^n$ terms of the series. It follows that $\lim_{n\to\infty} s_n = \infty$ and the Harmonic Series diverges.

**Note:** Mathematics is full of surprising, yet irrefutable, facts such as this. If you think the previous counter-example is cool, you should become a math major. :)

One final property of infinite series is that they obey the usual laws of addition, provided that the series are *convergent*.

**Linearity of Convergent Series**

If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

**(i)** $\sum_{n=1}^{\infty} ca_n = cA$ for any $c \in \mathbb{R}$ (constants pull out)

**(ii)** $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$ (the series of a sum equals the sum of the series)