MATH 136-02, 136-03 Calculus 2, Fall 2018

Sections 10.6 and 10.7: Power Series and Taylor Series

The final two sections on infinite series concern series where the terms being summed are functions of x, specifically power functions of the form $(x-c)^n$ for some constant c. These series, called power series, play an important role in applications of calculus since they are excellent approximations to more complicated functions such as e^x and $\sin x$.

Definition: Power Series

A **power series centered at** c is an infinite series of the form

$$F(x) \;=\; \sum_{n=0}^{\infty} a_n(x-c)^n \;=\; a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots .$$

Here, the **center** of the series is the constant c and the variable is x.

Example 1: The infinite series

$$F(x) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}(x-3)^n \;=\; 1 + (x-3) + \frac{1}{2!}(x-3)^2 + \frac{1}{3!}(x-3)^3 + \frac{1}{4!}(x-4)^4 + \cdots \tag{1}$$

is a power series centered at $c = 3$. Note that $0! = 1$ by convention. The series

$$G(x) \;=\; \sum_{n=1}^{\infty} \frac{1}{n}x^n \;=\; x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots \tag{2}$$

is a power series centered at $c = 0$. Note that although this series begins at $n = 1$, it is still considered a power series.

The main question when studying power series is to determine for which values of x the series converges. For example, in the series defined in equation (1) above, we have

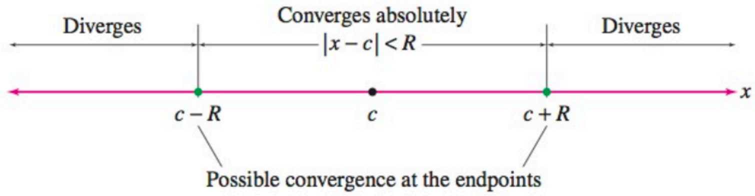
$$F(5) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}(5-3)^n \;=\; \sum_{n=0}^{\infty} \frac{2^n}{n!},$$

which converges by the ratio test (see Example 1 on the worksheet for Section 10.5). This allows us to define the function F at $x = 5$ to be the unique number that the infinite series converges to. On the other hand, in the series defined in equation (2) above, we have

$$G(1) \;=\; \sum_{n=1}^{\infty} \frac{1}{n}(1)^n \;=\; \sum_{n=1}^{\infty} \frac{1}{n} \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots,$$

which diverges because it is the Harmonic Series. Therefore, $G(1)$ is undefined.

While it may seem daunting to find the set of all x for which a given power series converges, it turns out that there is a unique value $R \geq 0$, called the **radius of convergence**, such that the power series converges absolutely for $|x-c| < R$ and diverges when $|x-c| > R$. In other words, for any power series centered at c, there is an **interval of convergence** centered at c of the form $c - R < x < c + R$



for which the power series converges. The series may or may not converge at the endpoints $x = c - R$ or $x = c + R$ (see figure above). If $R = 0$, then the series converges only when $x = c$. If $R = \infty$, then the power series converges for all x. The radius of convergence can be found using the ratio test.

Example 2: Use the ratio test to determine where $F(x) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}(x-3)^n$ converges.

Let $a_n = \frac{(x-3)^n}{n!}$. We apply the ratio test regarding x as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-3)^{n+1}}{(n+1)!}}{\frac{(x-3)^n}{n!}}\right| = \left|\frac{(x-3)^{n+1}}{(n+1)!}\right| \cdot \left|\frac{n!}{(x-3)^n}\right| = \frac{|(x-3)^n \cdot (x-3)|}{(n+1)\cdot n!} \cdot \frac{n!}{|(x-3)^n|} = \frac{|x-3|}{n+1}.$$

Then, since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-3|}{n+1} = |x-3| \cdot \lim_{n\to\infty}\frac{1}{n+1} = 0$, the power series converges for any x. The solution is $(-\infty, \infty)$ or $\mathbb{R}$. The radius of convergence is $R = \infty$.

Example 3: Use the ratio test to determine where $G(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}x^n$ converges.

Let $a_n = \frac{x^n}{n}$. We apply the ratio test regarding x as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \left|\frac{x^{n+1}}{n+1}\right| \cdot \left|\frac{n}{x^n}\right| = \frac{|x^n \cdot x|}{n+1} \cdot \frac{n}{|x^n|} = |x| \cdot \frac{n}{n+1}.$$

Then, since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|x| \cdot \frac{n}{n+1} = |x| \cdot \lim_{n\to\infty}\frac{n}{n+1} = |x|$, the power series converges for any x satisfying $|x| < 1$ by the ratio test. The radius of convergence is $R = 1$. This shows that the power series converges for $-1 < x < 1$ and diverges for $|x| > 1$. However, we must check the endpoints $x = 1$ and $x = -1$ directly to determine if the series converges at these points. We have already seen that $G(1) = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges since it is the Harmonic Series. On the other hand, notice that

$$G(-1) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - + \cdots$$

converges by the Alternating Series Test (it is -1 times the Alternating Harmonic Series). We conclude that the power series $G(x)$ converges for $-1 \leq x < 1$ or $[-1, 1)$.

Exercises: Find the interval of convergence for each of the following power series. Be sure to check the endpoints.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

2. $\displaystyle\sum_{n=1}^{\infty} (x-1)^n \frac{1}{n\, 3^n}$

3. $\displaystyle\sum_{n=1}^{\infty} (x-1)^n \frac{1}{n^2\, 3^n}$

Next we go in the opposite direction. How do we find the power series expansion for a given function $f(x)$? This material is covered in Section 10.7. We will assume that f is differentiable, that is, f has derivatives of all orders. It turns out that power series may be differentiated term by term over their interval of convergence (except at the endpoints). This leads to the following definition:

Definition: Taylor Series

The **Taylor series** expansion of a function $f(x)$ about $x = c$ is given by

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \;=\; f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots .$$

Here, $f^{(n)}$ represents the nth derivative of f. We define $f^0(c) = f(c)$.

The **Maclaurin series** for $f(x)$ is the special case where $c = 0$:

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \;=\; f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots . \tag{3}$$

We note that each of the series expansions above are only defined on the interval of convergence $c - R < x < c + R$ (and possibly at the endpoints), otherwise the expansions are invalid. For the sake of time, we will restrict our attention to Maclaurin series.

Example 4: Find the Maclaurin series for $f(x) = e^x$.

This is one of the most famous series in mathematics. Since $f'(x) = e^x, f''(x) = e^x, f'''(x) = e^x$ and more generally, $f^{(n)}(x) = e^x$, we see that $f^{(n)}(0) = 1$ for any n. Using formula (3), we have

$$\boxed{e^x \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!} \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .}$$

This series converges for any x, so the formula is valid for all real numbers. You should **memorize** this formula. Notice that if we differentiate both sides of the formula, we obtain a valid statement:

$$\frac{d}{dx}\left(e^x\right) \;=\; \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right) \;=\; 0 + 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots \;=\; e^x$$

Exercise 4: Show that the Maclaurin series expansion for $f(x) = \sin x$ is given by

$$\boxed{\sin x \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots .}$$

The Maclaurin series for $\cos x$ is given by

$$\boxed{\cos x \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots .}$$

The series expansions for both $\sin x$ and $\cos x$ converge for all real numbers. Notice that the series for $\sin x$ contains only odd powers while the series for $\cos x$ has only even powers. This agrees with the fact that $\sin x$ is an odd function, while $\cos x$ is even.

Another important example follows from the formula for the sum of an infinite geometric series:

$$\boxed{\frac{1}{1-x} \;=\; \sum_{n=0}^{\infty} x^n \;=\; 1 + x + x^2 + x^3 + x^4 + \cdots .}$$

Notice that the right-hand side is just a geometric series with first term $a = 1$ and ratio $r = x$. Thus, as long as $|r| = |x| < 1$, this series converges to the sum $S = \dfrac{a}{1-r} = \dfrac{1}{1-x}$.

Shortcuts for Finding Maclaurin Series

Given the Maclaurin series for a function $f(x)$, it is possible to find the Maclaurin series for $f(g(x)$ by replacing x with $g(x)$ in the series expansion.

Example 5: Find the Maclaurin series for $f(x) = e^{-x^2}$.

Replacing x by $-x^2$ in the Maclaurin series for e^x, we find

$$e^{-x^2} \;=\; \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \;=\; 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - + \cdots .$$

Since the original series converges for all x, so does the Maclaurin series for e^{-x^2}.

Exercises:

5. Find the Maclaurin series expansion for $f(x) = \cos(3x^2)$. State the first four terms of the series.

6. Find the Maclaurin series expansion for $f(x) = \dfrac{1}{1+3x}$. State the first four terms of the series and give the interval of convergence.

7. Using the Maclaurin series for e^x, $\sin x$, and $\cos x$, derive **Euler's formula**:

$$e^{i\theta} \;=\; \cos\theta + i\sin\theta\,.$$

Plug in $\theta = \pi$ to show that

$$\boxed{e^{i\pi} + 1 \;=\; 0\,,}$$

widely regarded as one of the most remarkable and elegant formulas ever discovered. *Hint:* Use the fact that $i^2 = -1, i^3 = -i, i^4 = 1, \ldots$.