MATH 136-02, 136-03 Calculus 2, Fall 2018

Sections 10.6 and 10.7: Power Series and Taylor Series

Solutions

Exercises: Find the interval of convergence for each of the following power series. Be sure to check the endpoints.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

   Answer: The center of the series is $c = 0$. Let $a_n = \frac{x^n}{2^n}$. We apply the ratio test regarding x as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}}\right| = \left|\frac{x^{n+1}}{2^{n+1}}\right| \cdot \left|\frac{2^n}{x^n}\right| = \frac{|x^n \cdot x|}{2^n \cdot 2} \cdot \frac{2^n}{|x^n|} = \frac{|x|}{2}.$$

   Then, since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{2} = \frac{|x|}{2}$, we solve $|x|/2 < 1$ to find where the power series converges (by the ratio test). This yields $|x| < 2$ and thus the radius of convergence is $R = 2$. The power series converges for $-2 < x < 2$ and diverges for $|x| > 2$. However, we must check the endpoints $x = 2$ and $x = -2$ directly to determine if the series converges at these points.

   Substituting $x = -2$ into the original series gives

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n,$$

   which diverges by the nth term test. Substituting $x = 2$ into the original series gives

$$\sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1,$$

   which also diverges by the nth term test. Therefore, the interval of convergence for the power series is $-2 < x < 2$ or $(-2, 2)$.

2. $\displaystyle\sum_{n=1}^{\infty} (x-1)^n \frac{1}{n\,3^n}$

   Answer: The center of the series is $c = 1$. Let $a_n = \frac{(x-1)^n}{n3^n}$. We apply the ratio test regarding x as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}}{\frac{(x-1)^n}{n3^n}}\right| = \left|\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}\right| \cdot \left|\frac{n3^n}{(x-1)^n}\right| = \frac{|(x-1)^n \cdot (x-1)|}{(n+1)3^n \cdot 3} \cdot \frac{n3^n}{|(x-1)^n|} = \frac{|x-1|}{3} \cdot \frac{n}{n+1}.$$

   Then, since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-1|}{3} \cdot \frac{n}{n+1} = \frac{|x-1|}{3} \cdot \lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-1|}{3} \cdot 1,$$

we solve $|x-1|/3 < 1$ to find where the power series converges (by the ratio test). This yields $|x-1| < 3$ and thus the radius of convergence is $R = 3$. The power series converges for $-2 < x < 4$ and diverges for $|x-1| > 3$. However, we must check the endpoints $x = -2$ and $x = 4$ directly to determine if the series converges at these points.

Substituting $x = -2$ into the original series gives

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test (it is -1 times the Alternating Harmonic Series). Substituting $x = 4$ into the original series gives

$$\sum_{n=1}^{\infty} \frac{3^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges since it is the Harmonic Series (a p-series with $p = 1$). Therefore, the interval of convergence for the power series is $-2 \le x < 4$ or $[-2, 4)$.

3. $\displaystyle\sum_{n=1}^{\infty} (x-1)^n \frac{1}{n^2\,3^n}$

Answer: Notice the similarity with the previous problem. The center of the series is $c = 1$. Let $a_n = \frac{(x-1)^n}{n^2 3^n}$. We apply the ratio test regarding x as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-1)^{n+1}}{(n+1)^2 3^{n+1}}}{\frac{(x-1)^n}{n^2 3^n}}\right| = \left|\frac{(x-1)^{n+1}}{(n+1)^2 3^{n+1}}\right| \cdot \left|\frac{n^2 3^n}{(x-1)^n}\right| = \frac{|(x-1)^n \cdot (x-1)|}{(n+1)^2 3^n \cdot 3} \cdot \frac{n^2 3^n}{|(x-1)^n|}$$

$$= \frac{|x-1|}{3} \cdot \frac{n^2}{n^2 + 2n + 1}.$$

Then, since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-1|}{3} \cdot \frac{n^2}{n^2+2n+1} = \frac{|x-1|}{3} \cdot \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \frac{|x-1|}{3} \cdot 1,$$

we solve $|x-1|/3 < 1$ to find where the power series converges (by the ratio test). This yields $|x-1| < 3$ and thus the radius of convergence is $R = 3$. The power series converges for $-2 < x < 4$ and diverges for $|x-1| > 3$. However, we must check the endpoints $x = -2$ and $x = 4$ directly to determine if the series converges at these points.

Substituting $x = -2$ into the original series gives

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges by the Alternating Series Test or by the Absolute Convergence Test. Substituting $x = 4$ into the original series gives

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2\,3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges since it is a p-series with $p = 2$. Therefore, the interval of convergence for the power series is $-2 \le x \le 4$ or $[-2, 4]$.

Exercise 4: Show that the Maclaurin series expansion for $f(x) = \sin x$ is given by

$$\boxed{\sin x \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots .}$$

Answer: Since $f(x) = \sin x$, we have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{iv}(x) = \sin x$, and then the pattern repeats. Thus, $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{iv}(0) = 0, f^{v}(0) = 1, f^{vi}(0) = 0, \ldots$. The pattern $0, 1, 0, -1$ repeats itself over and over again. Using the Maclaurin series formula, we find

$$\begin{aligned}
\sin x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots \\
&= x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{-1}{7!}x^7 + - \cdots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots .
\end{aligned}$$

Notice that the even powers vanish from the series because the even derivatives at $x = 0$ all evaluate to zero. Recalling that an odd integer can be written as $2n + 1$, we have $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

5. Find the Maclaurin series expansion for $f(x) = \cos(3x^2)$. State the first four terms of the series.

   Answer: Replacing x by $3x^2$ in the Maclaurin series for $\cos x$, we have

$$\begin{aligned}
\cos(3x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(3x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n} x^{4n}}{(2n)!} = 1 - \frac{9x^4}{2!} + \frac{81x^8}{4!} - \frac{729x^{12}}{6!} + - \cdots \\
&= 1 - \frac{9x^4}{2} + \frac{27x^8}{8} - \frac{81x^{12}}{80} + - \cdots .
\end{aligned}$$

6. Find the Maclaurin series expansion for $f(x) = \dfrac{1}{1+3x}$. State the first four terms of the series and give the interval of convergence.

   Answer: Replacing x by $-3x$ (notice the negative sign!) in the Maclaurin series for $\dfrac{1}{1-x}$, we have

$$\begin{aligned}
\frac{1}{1+3x} &= \sum_{n=0}^{\infty} (-3x)^n = 1 - 3x + (-3x)^2 + (-3x)^3 + (-3x)^4 + \cdots \\
&= 1 - 3x + 9x^2 - 27x^3 + 81x^4 - + \cdots .
\end{aligned}$$

   Notice that the series is geometric with ratio $r = -3x$ so the series converges for $|r| = |-3x| < 1$, which simplifies to $|x| < 1/3$. Thus the interval of convergence is $-1/3 < x < 1/3$.

7. Using the Maclaurin series for e^x, $\sin x$, and $\cos x$, derive **Euler's formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

Plug in $\theta = \pi$ to show that

$$\boxed{e^{i\pi} + 1 = 0\,,}$$

widely regarded as one of the most remarkable and elegant formulas ever discovered. *Hint:* Use the fact that $i^2 = -1, i^3 = -i, i^4 = 1, \ldots$.

Answer: Replacing x by $i\theta$ in the Maclaurin series for e^x and using the hint, we find

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} \cdots \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + - \cdots\right) \\
&= \cos\theta + i\sin\theta\,.
\end{aligned}$$

Plugging in $\theta = \pi$, we find $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i\cdot 0 = -1$. Therefore $e^{i\pi} + 1 = 0$.