MATH 135-08, 135-09 Calculus 1, Fall 2017

Using the First Derivative to Understand a Function

Worksheet for Section 4.3

Key Idea: This section focuses on using the first derivative to understand properties of a function such as where it is increasing or decreasing. The first derivative can also be used to determine whether a critical point is a local maximum, minimum, or neither.

One important fact from Calculus involving the first derivative is the **Mean Value Theorem (MVT)**. It states that if a function is differentiable on an interval, then there is a point somewhere in the interval where the derivative is equal to the slope of the line connecting the endpoints of the function (see Figure 1). The MVT is used to rigorously prove many results involving the first derivative.

Theorem 0.1 (The Mean Value Theorem) *Suppose that the function $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on (a,b). Then there exists a number $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

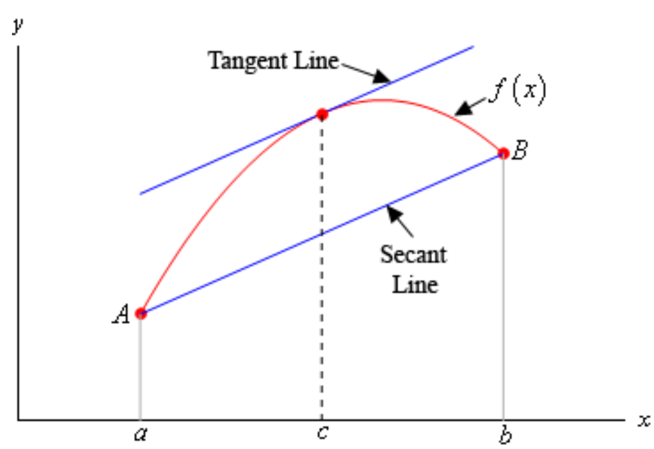

Figure 1: The Mean Value Theorem: a differentiable function on an interval (a,b) has at least one point $c \in (a,b)$ where $f'(c)$ equals the slope of the secant line between the endpoints of the interval.

Using the MVT, we can show that the first derivative indicates whether a function is increasing (positive slope), decreasing (negative slope), or neither (critical point; zero slope).

$$\boxed{\begin{array}{c} f'(x) > 0 \text{ for } x \in (a,b) \implies f \text{ is increasing on } (a,b) \\ f'(x) < 0 \text{ for } x \in (a,b) \implies f \text{ is decreasing on } (a,b) \\ f'(c) = 0 \implies c \text{ is a critical point of } f \end{array}}$$

By checking the sign of the first derivative to either side of a critical point, the first derivative can be used to determine the type of critical point. This is called the **first derivative test**. For instance, if $f'(4) = 0$, then $x = 4$ is a critical point. If $f' > 0$ for x slightly less than 4 and $f' < 0$ for x slightly greater than 4, we know that $x = 4$ is a local maximum (the function increases as x approaches 4, then begins to decrease.)

First Derivative Test: Suppose that $x = c$ is a critical point of f.

$f'(x)$ changes from $+$ to $-$ at c	$\Longrightarrow$	c is a local max
$f'(x)$ changes from $-$ to $+$ at c	$\Longrightarrow$	c is a local min

Note: If f' does not change sign at a critical point, then $x = c$ is neither a max nor a min.

It is helpful to draw a first derivative number line and indicate where f' is positive, negative, or 0.

Example 1: Let $f(x) = x^3 - 3x^2 - 45x + 5$. Find the critical points of f and use the first derivative test to classify each critical point as a local max, local min, or neither. On what interval(s) is f increasing? decreasing? Use this information to sketch a graph of f.

Example 2: For the function $g(x) = \dfrac{x^2 - 8x}{x + 1}$, find and simplify $g'(x)$. Then find the critical points and use the first derivative test to classify each one as a local max, local min, or neither.

Example 3: For each of the following functions, find the critical points and the intervals on which the function is increasing or decreasing. Then use the first derivative test to determine whether each critical point is a local max, local min, or neither.

(a) $F(x) = (x^2 - 2x)e^x$

(b) $h(x) = 15x^3 - x^5$