MATH 135-08, 135-09 Calculus 1, Fall 2017

The Derivative as a Function: Worksheet for Section 3.2

Recall that the derivative of a function at a point gives the slope of the tangent line at that point. In other words, $f'(a)$ represents the slope of the tangent line at $x = a$. In this section, we vary the point a, and treat the derivative as a function in its own right, the function $f'(x)$. The definition is the same as before, except that now we replace a by the variable x.

Definition 0.1 *The* **derivative function** $f'(x)$ *is given by*

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.} \tag{1}$$

The derivative function inputs the number x and outputs the slope of the tangent line to f at x. Thus, $f'(x)$ is essentially a **slope function**.

Important: Since it is defined in terms of a limit, the derivative may not exist at a given point. If $f'(a)$ exists, we say that f is **differentiable** at $x = a$. In general, $f'(a)$ does not exist if f has a corner, cusp, or vertical tangent line at the point $x = a$. For example, if $f(x) = |x|$, then $f'(0)$ does not exist because f has a corner at the origin. Similarly, $g'(0)$ does not exist for the function $g(x) = \sqrt{|x|}$ because g has a cusp at the origin.

Exercise 0.2 *Draw two different functions, $f(x)$ and $g(x)$, each with a point where the derivative does not exist.*

Theorem 0.3 *If $f(x)$ is differentiable at $x = a$, then it is also continuous there. However, the converse is not true: a function may be continuous at a point, but not differentiable there (e.g., $f(x) = |x|$ is continuous at $x = 0$, but not differentiable there).*

Leibniz Notation

There are many ways to write the derivative mathematically. One of the most popular is to use the notation introduced by Leibniz. If $y = f(x)$, then another way to write $f'(x)$ is $\frac{dy}{dx}$, which is read "the derivative of y with respect to x." This notation is useful for reminding us that the derivative is slope, so that

$$m \approx \frac{\Delta y}{\Delta x} \quad \text{suggests} \quad f'(x) = \frac{dy}{dx}.$$

Technically speaking, dy and dx are examples of differential one-forms, but you can just think of the d as representing the operation of taking the derivative. For example, the symbol $\frac{d}{dx}$ means differentiate (take the derivative) with respect to x. Thus,

$$\frac{d}{dx}(mx+b) \;=\; m$$

because the derivative of a linear function is just the slope of the line.

Useful Formulas Involving the Derivative

1. $\frac{d}{dx}(c) \;=\; 0$ (The derivative of a constant is zero.)
2. $\frac{d}{dx}(mx+b) \;=\; m$ (The derivative of a line is its slope.)
3. $\frac{d}{dx}(x^n) \;=\; nx^{n-1}$ for **any** real number n. (Power Rule)
4. $\frac{d}{dx}(cf(x)) \;=\; cf'(x)$ (Constants pull out.)
5. $\frac{d}{dx}(f(x)+g(x)) \;=\; f'(x)+g'(x)$ and $\frac{d}{dx}(f(x)-g(x)) \;=\; f'(x)-g'(x)$ (Linearity)
6. $\frac{d}{dx}(e^x) \;=\; e^x$ (The derivative of e^x is itself.)
7. $\frac{d}{dx}(b^x) \;=\; (\ln b)\cdot b^x$ (The derivative of an exponential function is a constant times itself.)

Each of the above formulas can be derived using the limit definition of the derivative. For instance, if $f(x) = mx + b$ (a linear function), then we would expect that $f'(x) = m$, since the slope of the tangent line is the same as the slope of the line itself, m. In terms of the limit definition, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{m(x+h)+b-(mx+b)}{h} \\ &= \lim_{h\to 0}\frac{mx+mh+b-mx-b}{h} \\ &= \lim_{h\to 0}\frac{mh}{h} \;=\; \lim_{h\to 0} m \;=\; m. \end{aligned}$$

Similarly, if $g(x) = x^3$, then we have

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} \\ &= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\ &= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h} \\ &= \lim_{h\to 0} 3x^2+3xh+h^2 \;=\; 3x^2, \end{aligned}$$

which verifies the power rule for $n = 3$.

Exercise 0.4 *Find each of the following derivatives using the power rule.*

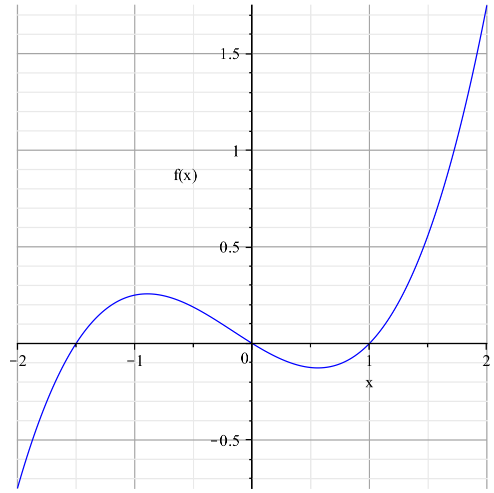
(a) $\frac{d}{dx}\left(x^{15}\right)$ **(b)** $\frac{d}{dx}\left(\frac{1}{x^4}\right)$ **(c)** $\frac{d}{dx}\left(\sqrt{x}\right)$ **(d)** $\frac{d}{dx}\left(x^{\pi}\right)$

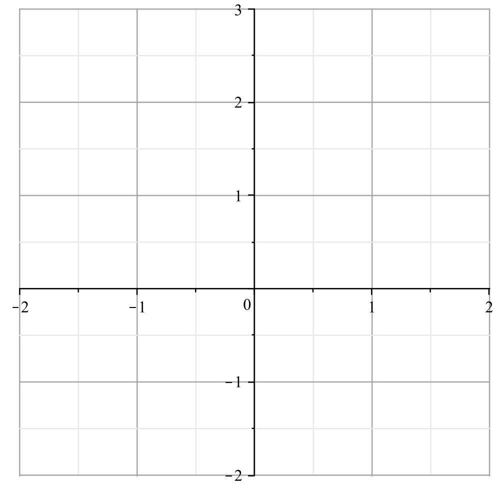
Exercise 0.5 *Find $g'(x)$ if $g(x) = 6\sqrt{x} - \frac{3}{x^3} + 5e^x - \pi^4$.*

Exercise 0.6 *If $f(x) = 4\sqrt[3]{x} + \frac{2}{3}x - \frac{8}{x}$, find the equation of the tangent line to f at the point $x = 8$.*

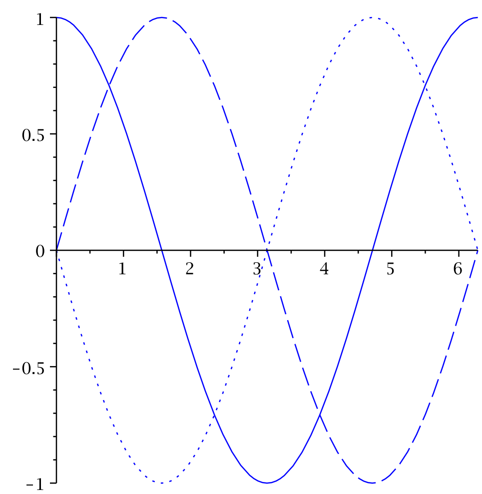
Exercise 0.7 *Using the limit definition of the derivative (Equation (1)), explain why $\frac{d}{dx}\left(e^x\right) = e^x$.*

Exercise 0.8 *Given the graph of $f(x)$ below, sketch the graph of the derivative function $f'(x)$ on the adjacent plot.* **Hint:** *Input x, output slope. Focus on the sign of the derivative first.*





Exercise 0.9 *The graph below shows three functions: $f(x), g(x)$, and $h(x)$. If $f'(x) = g(x)$ and $g'(x) = h(x)$, identify the graph that represents each function. Explain.*



Exercise 0.10 *If the graph of $g(t)$ is a parabola, what type of graph will $g'(t)$ be? Explain.*

Exercise 0.11 *If $z = e^t + t^e$, find $\dfrac{dz}{dt}$.*