MATH 135-08, 135-09 Calculus 1, Fall 2017

Trigonometric Limits: Worksheet for Section 2.6

This section focuses on two key limits involving $\sin x$ and $\cos x$ that are important for finding the slope of the tangent line to each function. These limits are proven through an important and intuitive theorem called the **Squeeze Theorem** (discussed previously in Section 2.3).

Theorem 0.1 (The Squeeze Theorem) *Suppose that $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at $x = a$) and that*

$$\lim_{x\to a} f(x) \;=\; \lim_{x\to a} h(x) \;=\; L.$$

Then $\lim_{x\to a} g(x) = L$.

The Squeeze Theorem states that if one function is "squeezed" between two others having a common limit, then the inner function takes on the same limit. It is best understood visually (see Figure 2 on p. 89 of the text).

Exercise 0.2 *Suppose that $g(x)$ satisfies $\cos x \le g(x) \le x^2 + 1$ for x-values near $x = 0$. Use the Squeeze Theorem to find $\lim_{x\to 0} g(x)$.*

Two important trigonometric limits are

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} \;=\; 1 \quad \text{and} \quad \lim_{x\to 0} \frac{1-\cos x}{x} \;=\; 0.} \tag{1}$$

These can be checked by using a calculator (set it to radians!) and plugging in values very close to 0. Note that each limit takes the form of $\frac{0}{0}$, an indeterminate form.

To prove the first limit in Equation (1), we use the fact that (see Figure 1)

$$\cos x \;\le\; \frac{\sin x}{x} \;\le\; 1 \qquad \text{for } -\pi/2 < x < \pi/2.$$

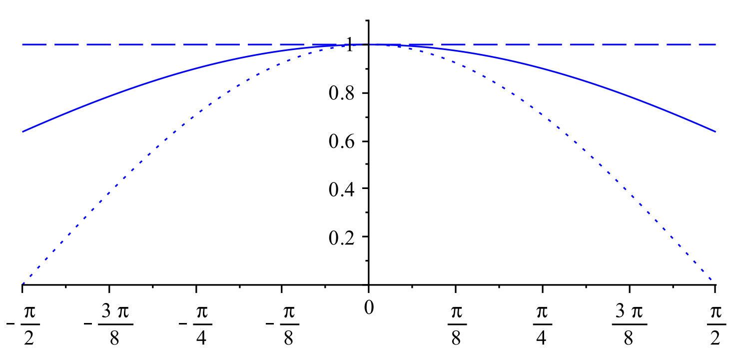

Figure 1: The graphs of the functions $y = 1$ (dashed), $y = \sin(x)/x$ (solid), and $y = \cos x$ (dotted).

Since $\lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} 1 = 1$, by the Squeeze Theorem, we have that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, as desired.

Exercise 0.3 *Using the limit* $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$*, show that* $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$.

Hint: *Multiply the top and bottom of* $\frac{1 - \cos\theta}{\theta}$ *by* $1 + \cos\theta$*, simplify, and break the fraction into the product of two fractions, one of which is* $\frac{\sin\theta}{\theta}$*. Then use the fact that the limit of a product is the product of the limits.*

Exercise 0.4 *Evaluate* $\lim_{x \to 0} \frac{\sin^2 x}{x^2}$. **Hint:** *The limit of the product equals the product of the limits.*

Exercise 0.5 *Use a calculator to evaluate* $\lim_{t \to 0} \frac{\sin(7t)}{t}$*. Then verify your answer by making the substitution* $x = 7t$.

Hint: *If* t *is tending toward 0, and* $x = 7t$*, then what is* x *approaching? Try and rewrite the limit using only the variable* x *so that the fraction* $\frac{\sin x}{x}$ *is present.*

Exercise 0.6 *Evaluate each limit.*

(a) $\lim_{\theta \to 0} \frac{\sin(9\theta)}{5\theta}$

(b) $\lim_{x \to 0} \frac{-4x}{\tan(7x)}$