MATH 135-08, 135-09 Calculus 1, Fall 2017

Limits at Infinity: Worksheet for Section 2.7

The expression $\lim_{x\to\infty} f(x)$ means to calculate the function values of f as x gets larger and larger, and see if they approach a limit. As with usual limits, the answer may be a real number L, ∞, $-\infty$, or the limit may not exist. For example,

$$\lim_{x\to\infty} x^2 = \infty$$

because as x gets larger, the value of x^2 gets even larger, and is therefore going to ∞. On the other hand, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

because as x gets larger, $1/x$ gets smaller and smaller. We say that the function $f(x) = 1/x$ has a **horizontal asymptote** at $y = 0$ because the graph of f approaches the horizontal line $y = 0$ as x tends to ∞. Horizontal asymptotes may also occur as x approaches $-\infty$ as well.

Note: The graph of a function *can* cross a horizontal asymptote, but as the values of x becomes larger and larger (or large and negative), the function should be heading closer and closer to the value of the asymptote.

The expression $\lim_{x\to-\infty} f(x)$ means to calculate the function values of f as x gets larger and larger, but negative. For example, we have

$$\lim_{x\to-\infty} x^2 = \infty, \quad \lim_{x\to-\infty} x^3 = -\infty, \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0.$$

Exercise 0.1 *Evaluate each of the following limits, if they exist.*

a. $\lim_{x\to\infty} e^{2x}$

b. $\lim_{x\to\infty} e^{-2x}$

c. $\lim_{x\to\infty} -x^4 + 3x^2 + 7$

d. $\lim_{x\to\infty} \sin x$

e. $\lim_{x\to-\infty} 2x^2 - 3x^3$

f. $\lim_{x\to\infty} \tan^{-1} x$

g. $\lim_{x\to-\infty} \tan^{-1} x$

Limits of Rational Functions

Consider the following limit: $\lim_{x\to\infty} \dfrac{3x^2+5x-2}{4x^2+7}$. To find it, we divide the top and bottom of the fraction by the **highest power in the denominator**, which in this case is x^2. This gives

$$\lim_{x\to\infty} \frac{3x^2+5x-2}{4x^2+7} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{5x}{x^2}-\frac{2}{x^2}}{\frac{4x^2}{x^2}+\frac{7}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{5}{x}-\frac{2}{x^2}}{4+\frac{7}{x^2}} = \frac{3}{4},$$

since each remaining fraction in the numerator and denominator is heading to 0 as x tends to ∞. The function has a horizontal asymptote at $y = 3/4$.

Exercise 0.2 *Evaluate* $\lim_{x\to\infty} \dfrac{6x^4-5x^3+2x}{4x^2-7x^4+1}$.

Exercise 0.3 *Evaluate* $\lim_{x\to\infty} \dfrac{6x^4-5x^3+2x}{4x^5-7x^4+1}$.

Exercise 0.4 *Evaluate* $\lim_{x\to\infty} \dfrac{\sqrt{25x^4+10}}{3x^2+1}$. **Hint:** *Ignore the* 10 *in the numerator. (Why is it ok to do this?)*

Exercise 0.5 *Evaluate* $\lim_{x\to\infty} \dfrac{e^x+3e^{-x}}{2e^x-e^{-x}}$. **Hint:** *What is the "highest" power in the denominator? Try dividing top and bottom of the fraction by it.*

Exercise 0.6 *Evaluate* $\lim_{x\to\infty} \tan^{-1}\left(\dfrac{5x^3-2x^2+3}{5x^3+9x^2-3x+\pi}\right)$.