MATH 135-08, 135-09 Calculus 1, Fall 2017

Inverse Functions: Worksheet for Section 1.5

The Inverse

One of the simplest ways to obtain a new function from an old one is to simply flip the domain and range. So if $f(2) = 7$, then the new function f^{-1}, called the **inverse of** f, has $f^{-1}(7) = 2$. The function f^{-1} simply maps each element in the range of f back to the element in the domain it came from; it "inverts" f.

However, there will be a problem if an element b in the range has two or more elements in the domain, say a_1 and a_2, that map to it (i.e., $f(a_1) = f(a_2) = b$). Then, the inverse of b would not be unique (it could be either a_1 or a_2), and f^{-1} would not be a function. To rectify this, we must assume that f is **one-to-one**, that is, each element in the range of f has one and only one pre-image that was sent to it in the domain.

For example, the function $f(x) = x^2$ is not one-to-one because both -2 and 2 are each sent to the same element in the range, $f(-2) = f(2) = 4$. This function fails the **horizontal line test** and is really two-to-one. However, if we restrict the domain to $x \geq 0$ (think of erasing the left half of the parabola), then f becomes one-to-one and now the inverse is actually a function. You know it already as $f^{-1}(x) = \sqrt{x}$. By definition (i.e., restricting the domain of x^2), $\sqrt{x}$ only spits out non-negative values.

Key Point: The only functions with well-defined inverses are those that are **one-to-one**. They must pass the **horizontal line test.**

Exercise 0.1 *Which of the following functions are one-to-one on their full domains? Draw a few examples to illustrate the difference between a one-to-one function and a function that is not one-to-one.*

a. $f(x) = \frac{2}{3}x - 7$ **b.** $F(x) = \frac{2}{3}$ **c.** $g(x) = (x-3)^2 + 5$ **d.** $G(x) = (x-3)^3 + 5$

e. $h(\theta) = 2\sin(3\theta)$ **f.** $H(\beta) = -4\tan(\beta)$ **g.** $i(t) = |t+1| - 4$ **h.** $I(t) = \sqrt{t+1} - 4$

Note that the notation for the inverse of f is not the usual exponent notation. In other words,

$$f^{-1} \neq \frac{1}{f}$$

The choice of -1 as the exponent is mathematical shorthand for **inverse**. Don't confuse this!

Based on the definition of the inverse of a function, the following formulas should make sense:

$$\boxed{f^{-1}(f(x)) \;=\; x \qquad \text{and} \qquad f(f^{-1}(x)) \;=\; x.} \tag{1}$$

Simply put, the inverse of f reverses what f does to x. Likewise, f reverses the action of f^{-1}, that is, the inverse of f^{-1} is just f. Note that the **domain of f^{-1} equals the range of f,** and vice-versa.

Graphically, if (x, y) is a point on the graph of f, then (y, x) is a point on the graph of f^{-1}. Thus, to obtain the graph of f^{-1} from the graph of f (or vice versa), just reflect the graph of f about the line $y = x$. This is also how one obtains an analytic formula for the inverse of a function: interchange the variables x and y and solve for y.

Exercise 0.2 *Find the inverse of the function $f(x) = 1 + \sqrt{2 + 3x}$. What should the domain of f^{-1} be in order to have f as its inverse?*

Inverse Trig Functions

The key to defining the inverse trig functions is to restrict the domains of the original trig functions in order to ensure that they are one-to-one. For example, the sine function is one-to-one on the domain $-\pi/2 \leq \theta \leq \pi/2$ (check the graph). By making this restriction, we then **define** the range of the inverse sine function (also called the **arcsine function**) to be $[-\pi/2, \pi/2]$. The domain of the inverse sine function is $[-1, 1]$ because this is precisely the range of the sine function.

Key Point: The inverse sine function, denoted $\sin^{-1}(x)$, inputs numbers between -1 and 1 and outputs angles between $-\pi/2$ and $\pi/2$. Thus, $\theta = \sin^{-1}(x)$ if and only if $\sin(\theta) = x$. (Go backwards!)

The domains and ranges of the other inverse trig functions are given below:

- $\cos^{-1}(x)$: Domain: $[-1, 1]$ Range: $[0, \pi]$
- $\tan^{-1}(x)$: Domain: $(-\infty, \infty)$ Range: $(-\pi/2, \pi/2)$
- $\cot^{-1}(x)$: Domain: $(-\infty, \infty)$ Range: $(0, \pi)$
- $\sec^{-1}(x)$: Domain: $(-\infty, -1] \cup [1, \infty)$ Range: $[0, \pi/2) \cup (\pi/2, \pi]$
- $\csc^{-1}(x)$: Domain: $(-\infty, -1] \cup [1, \infty)$ Range: $[-\pi/2, 0) \cup (0, \pi/2]$

Exercise 0.3 *Evaluate each of the following without using a calculator:*

a. $\sin^{-1}(1/\sqrt{2}) =$ ________ **b.** $\sin^{-1}(1) =$ ________ **c.** $\cos^{-1}(-\sqrt{3}/2) =$ ________

d. $\tan^{-1}(-1) =$ ________ **e.** $\sec^{-1}(2) =$ ________ **f.** $\csc^{-1}(-1) =$ ________

Exercise 0.4 *Explain why* $\pi/2$ *must be excluded from the range of* $\sec^{-1}(x)$*. In other words, why does the equation* $\sec^{-1}(x) = \pi/2$ *have no solution?*

Exercise 0.5 *Explain why* $\cos^{-1}(\cos(17\pi)) = \pi$ *and not* 17π*. What is* $\sin^{-1}(\sin(11\pi/3)$*?*

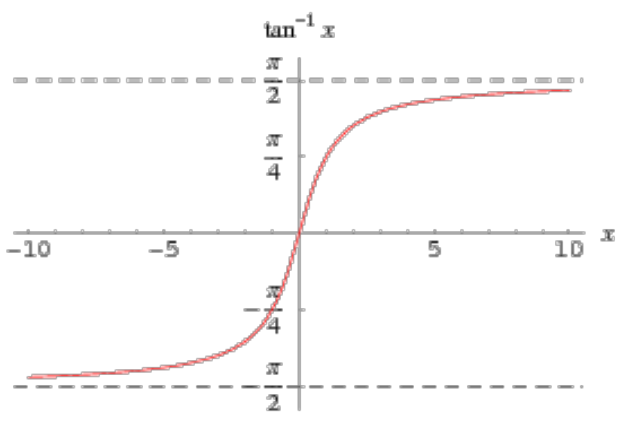

Figure 1: The graph of the inverse tangent function $y = \tan^{-1} x$ is cool. It maps the entire real line one-to-one and onto the open interval $(-\pi/2, \pi/2)$. Note the horizontal asymptotes at $y = -\pi/2$ and $y = \pi/2$.