

# MATH 135-01     Calculus 1

Exam #3 SOLUTIONS

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1. Calculate the derivative of each function. **Simplify** your answer as best as possible. (30 pts.)

(a)  $f(x) = \frac{3}{x^3} + 3^x + e^3$

**Answer:** First, to avoid using the quotient rule, we write  $f(x) = 3x^{-3} + 3^x + e^3$ . Then, using the power rule and the rule for differentiating an exponential function, we have

$$f'(x) = -9x^{-4} + 3^x \cdot \ln 3 = \frac{-9}{x^4} + (\ln 3) \cdot 3^x.$$

Note that  $e^3$  is just a constant, so its derivative is 0.

(b)  $g(x) = \frac{2 - 3x^2}{1 + 4x^2}$

**Answer:**  $g'(x) = \frac{-22x}{(1 + 4x^2)^2}.$

Using the quotient rule, we have

$$g'(x) = \frac{(1 + 4x^2)(-6x) - (2 - 3x^2)(8x)}{(1 + 4x^2)^2} = \frac{-6x - 24x^3 - 16x + 24x^3}{(1 + 4x^2)^2} = \frac{-22x}{(1 + 4x^2)^2}.$$

(c)  $F(t) = t^2 e^{\sqrt{4t+1}}$

**Answer:** Using the product and chain rules, we have

$$\begin{aligned} F'(t) &= 2t \cdot e^{\sqrt{4t+1}} + t^2 \cdot e^{\sqrt{4t+1}} \cdot \frac{1}{2}(4t+1)^{-1/2} \cdot 4 \\ &= e^{\sqrt{4t+1}} (2t + 2t^2(4t+1)^{-1/2}) \\ &= 2te^{\sqrt{4t+1}} \left( 1 + \frac{t}{\sqrt{4t+1}} \right). \end{aligned}$$

(d)  $G(\theta) = \ln(\ln(\sec \theta))$

**Answer:** Using the chain rule twice, we have

$$G'(\theta) = \frac{1}{\ln(\sec \theta)} \cdot \frac{1}{\sec \theta} \cdot \sec \theta \cdot \tan \theta = \frac{\tan \theta}{\ln(\sec \theta)}.$$

(e)  $y = x^{\tan x}$  *Hint:* Use logarithmic differentiation.

**Answer:** Take the natural log of both sides to obtain

$$\ln y = \ln x^{\tan x} = \tan x \cdot \ln x.$$

Next we differentiate both sides with respect to  $x$ , using implicit differentiation on the left-hand side and the product rule on the right-hand side. This gives

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sec^2 x \cdot \ln x + \tan x \cdot \frac{1}{x}.$$

Multiplying both sides by  $y$  and substituting in for  $y$  gives the answer

$$\frac{dy}{dx} = y \left( \sec^2 x \cdot \ln x + \frac{\tan x}{x} \right) = x^{\tan x} \left( \sec^2 x \cdot \ln x + \frac{\tan x}{x} \right).$$

2. For the equation below, use implicit differentiation to calculate  $dy/dx$ . (12 pts.)

$$xy^2 + e^{y^3} = \tan^{-1} x - \cos(4y)$$

**Answer:** Differentiating each side with respect to  $x$  and treating  $y = y(x)$  as a function of  $x$ , we have, by the chain rule,

$$1 \cdot y^2 + x \cdot 2y \cdot \frac{dy}{dx} + e^{y^3} \cdot 3y^2 \cdot \frac{dy}{dx} = \frac{1}{1+x^2} + \sin(4y) \cdot 4 \frac{dy}{dx}.$$

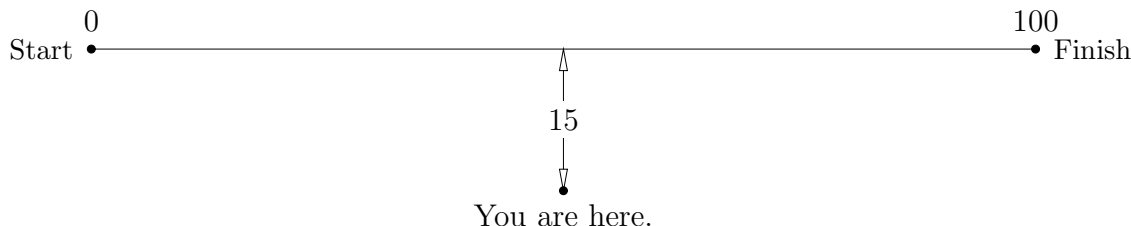
Grouping all terms with  $dy/dx$  together on one side of the equation yields

$$2xy \frac{dy}{dx} + 3y^2 e^{y^3} \frac{dy}{dx} - 4 \sin(4y) \frac{dy}{dx} = \frac{1}{1+x^2} - y^2$$

which gives, after factoring out the  $dy/dx$  on the left-hand side and dividing by the term in parentheses,

$$\frac{dy}{dx} = \frac{\frac{1}{1+x^2} - y^2}{2xy + 3y^2 e^{y^3} - 4 \sin(4y)}.$$

3. You are at a Holy Cross track meet watching the women's 100 meter dash, sitting in the bleachers 15 meters back from the halfway point of the race. Your friend Tori "the turbo" is running down the track at a constant speed of 8 meters per second. How fast is the distance between you and Tori changing when she is 75 meters into the race? Round your answer to two decimal places. (13 pts.)



**Answer:** 6.86 meters/sec

Draw a right triangle with the height equal to 15, the base equal to  $x$ , and the hypotenuse equal to  $z$ . The right angle of this triangle occurs at the midway point of the race, 50 meters along the track. The hypotenuse  $z$  is the distance between you and Tori. Both the quantities  $x$  and  $z$  are changing over time, but the distance 15 is fixed because you are not moving. When Tori is 75 meters into the race, we have  $x = 25$ . We want to find  $dz/dt$  when  $x = 25$ .

By the Pythagorean Theorem, we have  $x^2 + 15^2 = z^2$ . Differentiating this equation with respect to  $t$  yields  $2x \cdot \frac{dx}{dt} = 2z \cdot \frac{dz}{dt}$ , which simplifies to

$$\frac{dz}{dt} = \frac{x}{z} \cdot \frac{dx}{dt}.$$

We are given that  $dx/dt = 8$ . To find  $z$  when  $x = 25$ , use the Pythagorean Theorem to obtain  $z = \sqrt{25^2 + 15^2} = \sqrt{850} = 5\sqrt{34} \approx 29.155$ . Therefore, we find that

$$\frac{dz}{dt} = \frac{25}{5\sqrt{34}} \cdot 8 = \frac{40}{\sqrt{34}} \approx 6.86 \text{ meters/sec.}$$

4. Find the linearization  $L(x)$  of the function  $f(x) = \sqrt[3]{x}$  at the point  $a = 64$ . Use the linearization to approximate the value of  $\sqrt[3]{67}$  (to four decimal places) and then use a calculator to compute the percentage error in your approximation (to four decimal places). (12 pts.)

**Answer:** We use the formula  $L(x) = f(a) + f'(a)(x - a)$  with  $a = 64$ . Since  $f(x) = x^{1/3}$ , we have that  $f'(x) = \frac{1}{3}x^{-2/3}$ . Thus,  $f(64) = \sqrt[3]{64} = 4$  and

$$f'(64) = \frac{1}{3} \cdot 64^{-2/3} = \frac{1}{3} \cdot \frac{1}{64^{2/3}} = \frac{1}{3} \cdot \frac{1}{16} = \frac{1}{48}.$$

It follows that

$$L(x) = 4 + \frac{1}{48}(x - 64) = \frac{1}{48}x + \frac{8}{3}.$$

To approximate  $\sqrt[3]{67}$ , we plug in  $x = 67$  into our linearization:

$$L(67) = 4 + \frac{3}{48} = 4 + \frac{1}{16} = \frac{65}{16} = 4.0625.$$

The percentage error is then

$$100 \cdot \left| \frac{4.0625 - \sqrt[3]{67}}{\sqrt[3]{67}} \right| \approx 0.0234\%.$$

5. Find the absolute maximum and absolute minimum of the function

$$g(\theta) = \theta - 2 \sin \theta$$

over the interval  $[0, 2\pi]$ . Give the maximum and minimum function values (exact numbers as well as decimals rounded to three places), and the  $\theta$ -values (in radians) where they occur. (13 pts.)

**Answer:** The absolute minimum is  $\pi/3 - \sqrt{3} \approx -0.685$  at  $\theta = \pi/3$  and the absolute maximum is  $5\pi/3 + \sqrt{3} \approx 6.968$  at  $\theta = 5\pi/3$ .

First, we find the critical points of  $g$  on the interval  $[0, 2\pi]$ . We have  $g'(\theta) = 1 - 2 \cos \theta$ . Solving  $g'(\theta) = 0$  gives  $\cos \theta = 1/2$ , so  $\theta = \pi/3$  and  $\theta = 5\pi/3$  are the critical points.

Plugging the endpoints and the critical points into the function  $g$  gives

$$\begin{aligned} g(0) &= 0 \\ g(2\pi) &= 2\pi \approx 6.283 \\ g(\pi/3) &= \frac{\pi}{3} - \sqrt{3} \approx -0.685 \\ g(5\pi/3) &= \frac{5\pi}{3} + \sqrt{3} \approx 6.968. \end{aligned}$$

Thus, the absolute minimum occurs at  $\theta = \pi/3$  and the absolute maximum occurs at  $\theta = 5\pi/3$ .

6. Some final conceptual questions. (20 pts.)

- (a) When describing the recent price of VK stock, a market analyst states, “Although the price of VK stock continues to decline, it is declining at a slower rate. It might **not** be wise to completely sell off the stock.” If  $p(t)$  is the price of VK stock at time  $t$ , what are the signs (positive, negative, or zero) of  $p'(t)$  and  $p''(t)$ ?

**Answer:**  $p'(t) < 0$  and  $p''(t) > 0$ .

Since the price is decreasing, we have  $p'(t) < 0$ . But since the decline is slowing down, the slopes are getting less negative, that is, the derivative is increasing and the curve is concave up. Thus,  $p''(t) > 0$ .

- (b) The total dollar cost of producing  $x$  high-definition television sets is given by the function

$$C(x) = 300 - 100x - 0.2x^2 + 0.002x^3.$$

Find the marginal cost function and use it to *estimate* the cost of producing the 251st television set.

**Answer:** The marginal cost function is  $C'(x) = -100 - 0.4x + 0.006x^2$ . Recall that the marginal cost function estimates the cost of producing the next item (the  $(x+1)$ st item). Thus, to estimate the cost of producing the 251st television set, we compute  $C'(250) = \$175$ .

(c) Suppose that  $G(x) = f(x^2)$  and that  $f'(9) = 1/2$ ,  $f''(9) = -5$ . Find  $G''(3)$ .

**Answer:**  $-179$ .

By the chain rule, we have  $G'(x) = f'(x^2) \cdot 2x$ . To find  $G''(x)$  we use the product rule and the chain rule. We have

$$G''(x) = f''(x^2) \cdot 2x \cdot 2x + f'(x^2) \cdot 2 = 4x^2 \cdot f''(x^2) + 2f'(x^2).$$

Plugging  $x = 3$  into the last equation gives

$$G''(3) = 4 \cdot 9 \cdot f''(9) + 2 \cdot f'(9) = 36 \cdot (-5) + 2 \cdot \frac{1}{2} = -180 + 1 = -179.$$