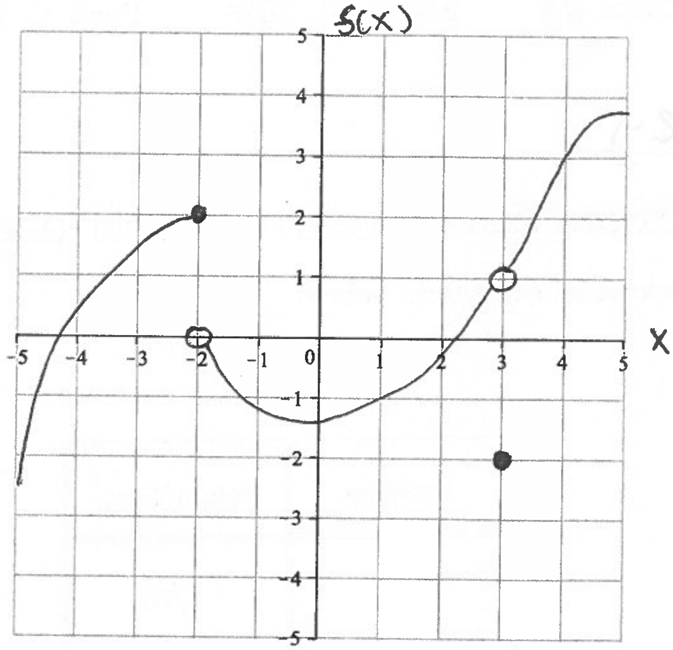
MATH 135-01 Calculus I

Exam #2 SOLUTIONS **March 31, 2016** **Prof. G. Roberts**

1. The graph of $f(x)$ is shown below. Use it to answer the following questions. (15 pts.)



(a) State the precise mathematical definition for a function $f(x)$ to be **continuous** at the point $x = a$.

Answer: $\lim_{x \to a} f(x) = f(a)$. There are three elements to this definition: (1) The function value $f(a)$ must exist; (2) The limit of the function at $x = a$ must exist; and (3) The function value and limit value must be equal to each other.

(b) Using your definition from part **(a)**, give a mathematical explanation as to why f is not continuous at $x = -2$.

Answer: $\lim_{x \to -2} f(x)$ does not exist (the left- and right-hand limits are different, causing a jump discontinuity).

(c) Is f left-continuous, right-continuous, or neither at $x = -2$? Explain.

Answer: f is left-continuous at $x = -2$ because the function value equals the left-hand limit, that is, $\lim_{x \to -2^-} f(x) = f(-2) = 2$.

(d) Evaluate each of the following:

(i) $\lim_{x \to 3} f(x) = 1$ (even though there is a hole in the graph, the limit still exists)

(ii) $f(3) = -2$

(e) How should $f(3)$ be **redefined** to remove the discontinuity at $x = 3$?

Answer: Set $f(3) = 1$. Since the limit of f at $x = 3$ is 1, to be continuous, we should redefine $f(3)$ to be 1. This plugs the hole in the graph, making the function continuous.

2. Evaluate each of the following limits, if they exist. Note that ∞ or $-\infty$ are acceptable answers. You must show work (e.g., algebra) to receive full credit. (5 pts. each)

   (a) $\lim_{x\to\pi} \sqrt{2\cos x + 5}$

   Answer: $\sqrt{3}$. Using the fact that the function is continuous at $x = \pi$, we simply plug in $x = \pi$ to obtain $\sqrt{2\cos\pi + 5} = \sqrt{2(-1) + 5} = \sqrt{3}$.

   (b) $\lim_{x\to 4} \dfrac{2x^2 - 7x - 4}{x^2 - 16}$

   Answer: 9/8. Factor and cancel. We have

$$\lim_{x\to 4} \frac{2x^2 - 7x - 4}{x^2 - 16} = \lim_{x\to 4} \frac{(2x+1)(x-4)}{(x+4)(x-4)} = \lim_{x\to 4} \frac{2x+1}{x+4} = \frac{2(4)+1}{4+4} = \frac{9}{8}.$$

   (c) $\lim_{\theta\to 0} \dfrac{\cos(3\theta)\cdot\sin\theta}{\theta}$

   Answer: 1. Rewrite the limit as a product of two fractions and use the fact that the limit of a product equals the product of the limits. We have

$$\lim_{\theta\to 0} \frac{\cos(3\theta)\cdot\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{\cos(3\theta)}{1}\cdot\frac{\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{\cos(3\theta)}{1}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\cdot 1 = 1,$$

   since $\cos(3\cdot 0) = \cos 0 = 1$.

   (d) $\lim_{x\to 2} \left(\dfrac{1}{2(x-2)} - \dfrac{2}{x^2-4}\right)$

   Answer: 1/8. Add the fractions and then simplify. Note that the least common denominator is $2(x-2)(x+2)$. We have

$$\begin{aligned}
\lim_{x\to 2} \left(\frac{1}{2(x-2)} - \frac{2}{x^2-4}\right) &= \lim_{x\to 2} \left(\frac{1}{2(x-2)} - \frac{2}{(x-2)(x+2)}\right) \\
&= \lim_{x\to 2} \left(\frac{x+2}{2(x-2)(x+2)} - \frac{2\cdot 2}{2(x-2)(x+2)}\right) \\
&= \lim_{x\to 2} \frac{x+2-4}{2(x-2)(x+2)} \\
&= \lim_{x\to 2} \frac{x-2}{2(x-2)(x+2)} \\
&= \lim_{x\to 2} \frac{1}{2(x+2)} \\
&= \frac{1}{2\cdot 4} = \frac{1}{8}.
\end{aligned}$$

   (e) $\lim_{x\to\infty} \tan^{-1}\left(\dfrac{1-x^4}{3x^3 - 4x^2 + 5}\right)$

   Answer: $-\pi/2$. First take the limit of the fraction inside the parentheses. Since the numerator has the higher power, and since the leading coefficients of the numerator and denominator have opposite signs, the fraction is heading toward $-\infty$. Now we need to compute $\tan^{-1}(-\infty)$, or more specifically, $\lim_{x\to -\infty} \tan^{-1} x$. Recall from its graph that $\tan^{-1} x$ has a horizontal asymptote to the left at $y = -\pi/2$. Thus, the limit we seek is equal to $-\pi/2$.

3. Use the Intermediate Value Theorem to prove that the equation $e^x = \cos(2x) + 1$ has a solution in the interval $[0, \pi/4]$. (10 pts.)

   Answer: First, we define the function $f(x) = e^x - \cos(2x) - 1$. We would like to find a number c between 0 and $\pi/4$ such that $f(c) = 0$, as then we would have $e^c - \cos(2c) - 1 = 0$, which is equivalent to $e^c = \cos(2c) + 1$ (the equation we would like to solve). Note that f is a continuous function since it is the difference of two continuous functions, an exponential function and a trig function.

   We compute that $f(0) = e^0 - \cos(0) - 1 = -1 < 0$ while $f(\pi/4) = e^{\pi/4} - \cos(\pi/2) - 1 = e^{\pi/4} - 1 \approx 1.1933 > 0$. By the Intermediate Value Theorem, since $f(0) < 0$ and $f(\pi/4) > 0$, there exists a number c between 0 and $\pi/4$ such that $f(c) = 0$, as desired.

4. **(a)** State the two limit definitions for the derivative of a function $f(x)$ at the point $x = a$. (4 pts.)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

   or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   (b) Using one of your limit definitions from part **(a)**, find $f'(2)$ if $f(x) = \sqrt{5x - 1}$. (9 pts.)

   Answer: $f'(2) = 5/6$.

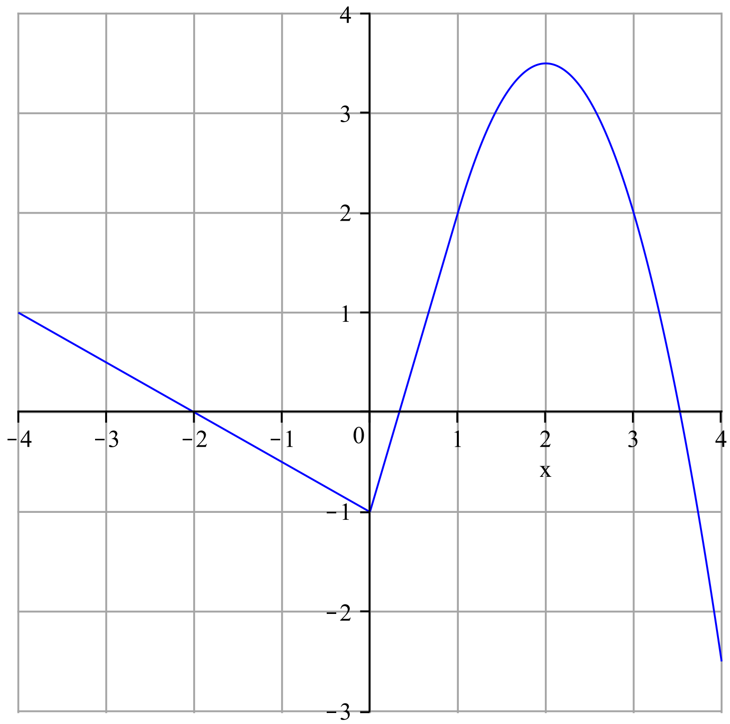
   Method 1: Using the $\lim_{h \to 0}$ definition, we have

$$\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{5(2+h) - 1} - 3}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{10 + 5h - 1} - 3}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{5h + 9} - 3}{h} \\
&= \lim_{h \to 0} \frac{(\sqrt{5h + 9} - 3)}{h} \cdot \frac{(\sqrt{5h + 9} + 3)}{(\sqrt{5h + 9} + 3)} \\
&= \lim_{h \to 0} \frac{5h + 9 - 9}{h(\sqrt{5h + 9} + 3)} \\
&= \lim_{h \to 0} \frac{5}{(\sqrt{5h + 9} + 3)} \\
&= \frac{5}{\sqrt{9} + 3} = \frac{5}{6}.
\end{aligned}$$

Method 2: Using the $\lim\limits_{x\to a}$ definition, we have

$$\begin{aligned}
f'(2) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\\
&= \lim_{x\to 2}\frac{\sqrt{5x-1}-3}{x-2}\\
&= \lim_{x\to 2}\frac{(\sqrt{5x-1}-3)}{x-2}\cdot\frac{(\sqrt{5x-1}+3)}{(\sqrt{5x-1}+3)}\\
&= \lim_{x\to 2}\frac{5x-1-9}{(x-2)(\sqrt{5x-1}+3)}\\
&= \lim_{x\to 2}\frac{5x-10}{(x-2)(\sqrt{5x-1}+3)}\\
&= \lim_{x\to 2}\frac{5(x-2)}{(x-2)(\sqrt{5x-1}+3)}\\
&= \lim_{x\to 2}\frac{5}{\sqrt{5x-1}+3}\\
&= \frac{5}{\sqrt{10-1}+3} \;=\; \frac{5}{6}.
\end{aligned}$$

5. The graph of $f(x)$ is shown below. Determine (estimate if necessary) the following values of the derivative: (10 pts.)



(a) $f'(-2)$

Answer: $-1/2$. The derivative $f'(x)$ is the **slope function**. So $f'(-2)$ is the slope of the tangent line at the point $x = -2$. The graph of f is a line of slope $m = -1/2$ at $x = -2$, so $f'(-2) = -1/2$.

(b) $f'(0)$

Answer: Does not exist. Since the graph of f has a corner at $x = 0$, the derivative does not exist.

(c) $f'(1/2)$

Answer: 3. At $x = 1/2$, the function is linear, with a slope of $m = 3$.

(d) $f'(2)$

Answer: 0. At $x = 2$, the function has a horizontal tangent line.

(e) $f'(3)$

Answer: -3. Estimating the slope of the tangent line at $x = 3$ gives a slope approximately equal to -3.

6. **Calculus Potpourri:** You must show your work to receive any partial credit. (27 pts.)

(a) If $\lim_{x\to -2} f(x) = 5$, find $\lim_{x\to -2} \dfrac{f(x)+1}{x^3}$.

Answer: $-3/4$. Using the limit laws, we have

$$\lim_{x\to -2} \frac{f(x)+1}{x^3} = \frac{\lim_{x\to -2}(f(x)+1)}{\lim_{x\to -2} x^3} = \frac{5+1}{(-2)^3} = \frac{6}{-8} = -\frac{3}{4}.$$

(b) If $6x - 8 \le g(x) \le 3x^2 - 5$ for all x, find $\lim_{x \to 1} g(x)$.

Answer: -2. Using the Squeeze Theorem, since $\lim_{x\to 1} 6x - 8 = -2$ and $\lim_{x\to 1} 3x^2 - 5 = -2$, we have that $\lim_{x\to 1} g(x) = -2$.

(c) Find any horizontal asymptotes for the function $y(t) = \dfrac{8t - 5t^3 + 9t^7}{\pi + 4t^5 - 3t^7}$.

Answer: $y = -3$. To find horizontal asymptotes, we need to compute $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$. Since the highest power in the numerator and denominator is the same, namely t^7, we simply read off the coefficients of this term. Both limits give the same value of $-9/3 = -3$.

(d) Find and simplify $F'(x)$ if $F(x) = 5e^x - \dfrac{3}{x^5} + \pi^3$.

Answer: $F'(x) = 5(e^x + 3x^{-6})$. First, rewrite F as $F(x) = 5e^x - 3x^{-5} + \pi^3$. Then, using the power rule, we have $F'(x) = 5e^x + 15x^{-6}$. Note that π^3 is just a constant, so its derivative is 0. Factoring out the 5 gives $F'(x) = 5(e^x + 3x^{-6})$.

(e) Find the equation of the tangent line (in slope-intercept form) to $f(x) = 6\sqrt{x} - \frac{1}{4}x^2$ at the point $a = 4$.

Answer: $y = -\frac{1}{2}x + 10$. First we need to find the derivative of f in order to find the slope of the tangent line. We rewrite f as $f(x) = 6x^{1/2} - \frac{1}{4}x^2$ and then apply the power rule. This gives $f'(x) = 3x^{-1/2} - \frac{1}{2}x$. The slope of the tangent line at $a = 4$ is $f'(4) = 3/2 - 2 = -1/2$.

Next we need to find a point on the line. If $x = 4$, then $f(4) = 6 \cdot 2 - \frac{1}{4} \cdot 16 = 8$. So $(4, 8)$ is on our line. Plugging in $x = 4, y = 8$ and $m = -1/2$ into $y = mx + b$ gives $8 = -\frac{1}{2} \cdot 4 + b$ so that $b = 10$. Hence, the equation of the tangent line is $y = -\frac{1}{2}x + 10$.