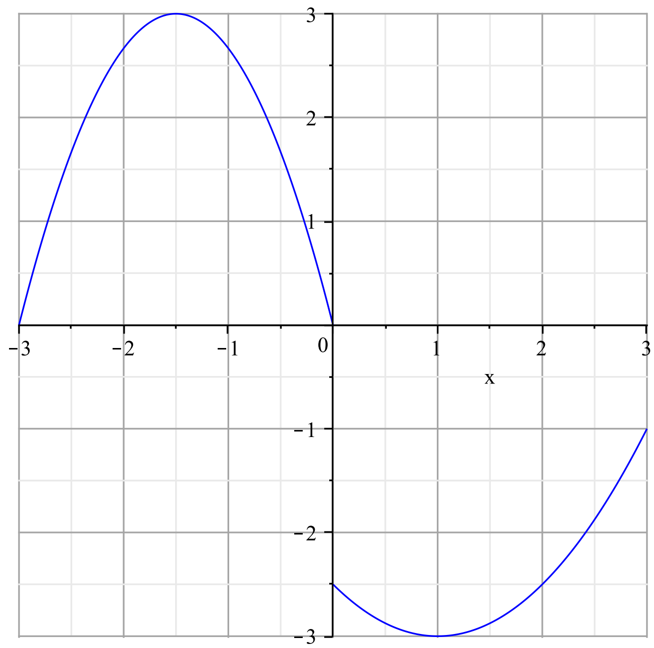
MATH 135-01 Calculus 1

Exam #1 SOLUTIONS **February 25, 2016** **Prof. G. Roberts**

1. The **ENTIRE** graph of $f(x)$ is shown below. Use it to answer each of the following questions: (18 pts.)



(a) What is the range of f? You may express your answer in interval notation or using inequalities.

Answer: $[-3,-1] \cup [0,3]$. The range is found by projecting the graph onto the y-axis. Note that there is a gap in the function and that no y-values between -1 and 0 have pre-images in the domain.

(b) Is f a one-to-one function? Explain.

Answer: No, it fails the horizontal line test. Specifically, there are output y-values of the function that have more than one pre-image x in the domain. For example, $f(-3) = f(0) = 0$.

(c) Suppose that $g(x) = e^{-x}$. Find the value of $f(g(0))$.

Answer: -3

$$f(g(0)) \;=\; f(e^0) \;=\; f(1) \;=\; -3.$$

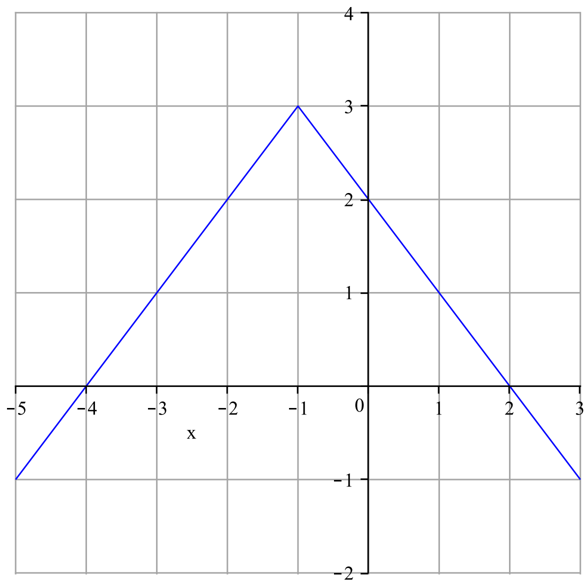
(d) Evaluate each of the following limits:

(i) $\lim_{x \to 0+} f(x) = -2.5$ **(ii)** $\lim_{x \to 0-} f(x) = 0$

(iii) $\lim_{x \to 0} f(x)$ does not exist because the left- and right-hand limits are not equal.

2. Below are the graphs of two functions, $f(x)$ and $g(\theta)$. Each graph was obtained by performing some geometric transformation(s) (stretching, shifting, reflecting, etc.) to a well known mathematical function. Find a formula for each function. (12 pts.)

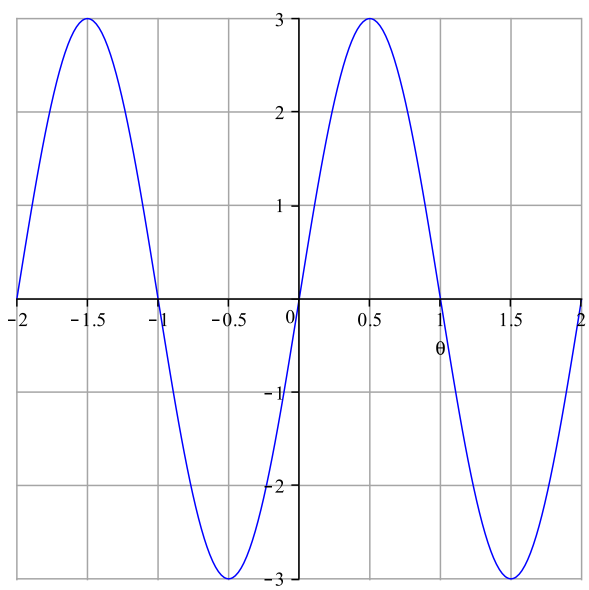
(a)



Answer: $f(x) = -|x+1| + 3$

The graph is that of the absolute value function $y = |x|$ after it has been reflected about the x-axis (multiply by -1), shifted left by 1 unit (replace x by $x+1$), and shifted up by three units (add 3).

(b)



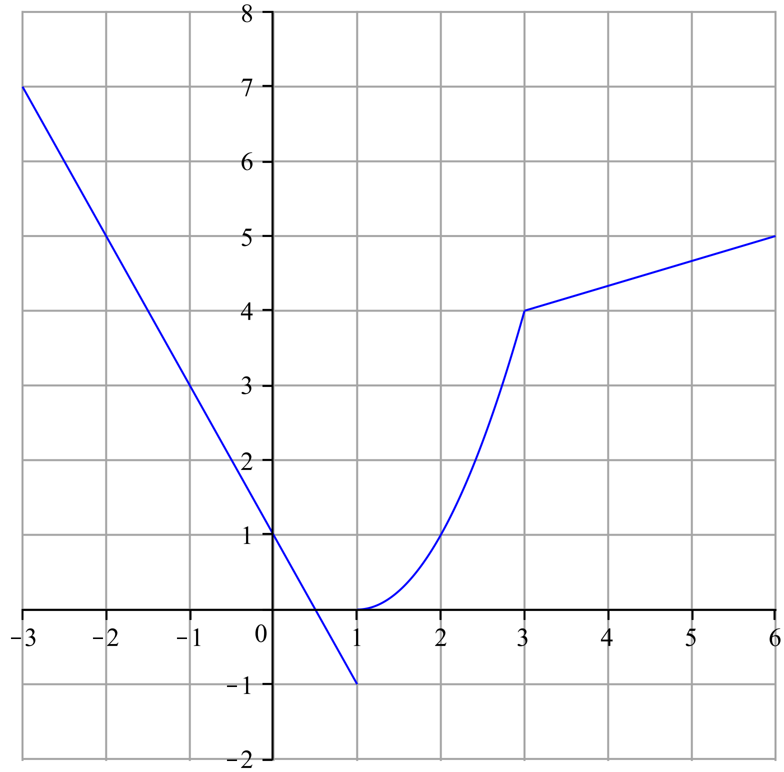
Answer: $g(\theta) = 3\sin(\pi\theta)$

This graph is a sine function with amplitude 3 ($a = 3$) and period 2. It has been stretched vertically by a factor of 3 and compressed horizontally by a factor of π. To see that $b = \pi$, recall that the period of $a\sin(b\theta)$ is given by $2\pi/b$. Solving $2\pi/b = 2$ yields $b = \pi$.

3. Carefully sketch the graph of the following piecewise function on the axes below. (10 pts.)

$$f(x) = \begin{cases} 1 - 2x & \text{if } -3 \le x < 1 \\ (x-1)^2 & \text{if } 1 \le x < 3 \\ \frac{1}{3}(x+9) & \text{if } 3 \le x \le 6 \end{cases}$$

Answer:



The first piece of the graph is a line of slope -2 and y-intercept 1. It runs all the way to the point $(1, -1)$, but does NOT include this point because $x < 1$. The second piece of the graph is a parabola opening up with vertex at $(1, 0)$. This point IS included on the graph because $1 \le x$. The final piece of the graph is a line of slope $1/3$ that starts at the point $(3, 4)$ and ends at the point $(6, 5)$. Since the parabola goes through the point $(3, 4)$, the second line and the parabola connect exactly (the graph is continuous at $x = 3$).

4. **Trig is Fun** (20 pts.)

(a) State the domain and range of the function $h(x) = \sin^{-1}(x)$.

Domain: $[-1, 1]$ Range: $[-\pi/2, \pi/2]$

Answer: The domain of $\sin^{-1}(x)$ is equal to the range of its inverse function, $\sin x$. Since the sine of an angle is the y-coordinate on the unit circle, then it is always between -1 and 1. The range of $\sin^{-1}(x)$ is defined to be the angles between $-\pi/2$ and $\pi/2$. This interval is chosen in order to make the sine function one-to-one (recall that a function must be one-to-one in order to have an inverse.)

(b) Find all angles θ between 0 and 2π that satisfy $\cos\theta = -1/\sqrt{2}$. Give your answer(s) in radians.

Answer: $3\pi/4, 5\pi/4$. First, we find the reference angle β, so that $\cos\beta = 1/\sqrt{2}$. Using a 45–45–90 right triangle, or from memory, $\beta = 45° = \pi/4$. Since $\cos\theta < 0$, we must chose θ to be in the second or third quadrant. Thus, the solution is $\theta = \pi - \pi/4 = 3\pi/4$ and $\theta = \pi + \pi/4 = 5\pi/4$.

(c) Suppose that $\tan\theta = -4/3$ and that $-\pi/2 \le \theta \le 0$. Find the values of $\cos\theta$ and $\csc\theta$.

Answer: $\cos\theta = 3/5$ and $\csc\theta = -5/4$.

Using SOH-CAH-TOA, draw a right triangle with sides 3 and 4, with the angle θ across from the side of length 4. By the Pythagorean Theorem, the hypotenuse has length 5. It follows that $\cos\theta = 3/5$ (adj./hyp.) and $\sin\theta = -4/5$ (opp./hyp.; negative because θ is in the fourth quadrant). Since $\csc\theta = 1/\sin\theta$, we have $\csc\theta = -5/4$.

(d) Evaluate each of the following limits (∞ and $-\infty$ are acceptable answers):

(i) $\lim_{x\to 0} \dfrac{4\sin x}{x} = 4$ **(ii)** $\lim_{t\to \frac{\pi}{2}^+} \tan t = -\infty$

Answer: Use a calculator to evaluate the first limit by plugging in x-values very close, but **not** equal to 0. **Note: You must always set your calculator to radians when using trig functions.** Radians are real, physical units and are **always** assumed to be the units for the angles we input into trig functions.

The second limit can be obtained with a calculator or by recalling the graph of $\tan t$. It has a vertical asymptote at $t = \pi/2$ and the graph approaches $-\infty$ as you approach the asymptote from the right.

5. **Average and Instantaneous Velocity** (12 pts.)

(a) Suppose that $s(t) = \dfrac{5}{2}t^2 - 2t$ represents the distance in feet a ball has traveled after t seconds. Compute the average velocity of the ball over the intervals $[1, 3]$ and $[1, 2]$ (give the correct units).

Answer: For $[1, 3]$, the average velocity is 8 feet per second. For $[1, 2]$, the average velocity is 5.5 feet per second.

Using the formula average velocity is $(s(t_2) - s(t_1))/(t_2 - t_1)$, we compute the average velocities to be

$$\frac{s(3) - s(1)}{3 - 1} = \frac{(45/2 - 6) - (5/2 - 2)}{3 - 1} = \frac{16}{2} = 8 \text{ ft/sec}$$

and

$$\frac{s(2) - s(1)}{2 - 1} = \frac{(10 - 4) - (5/2 - 2)}{2 - 1} = \frac{5.5}{1} = 5.5 \text{ ft/sec.}$$

(b) Find the instantaneous velocity of the ball at $t = 1$. Be sure to show your work.

Answer: 3 feet per second

The instantaneous velocity is defined to be the limit of the average velocities over smaller and smaller intervals around $t = 1$. Computing the average velocity on the interval $[1, 1.0001]$ gives 3.00025 ft/sec while computing the average velocity over $[0.9999, 1]$ gives 2.99975 ft/sec. It follows that the instantaneous velocity is 3 ft/sec.

6. **Calculus Potpourri** (28 pts.)

(a) The function $g(x) = \cos x + |x| + 5$ is <u>even</u>.

(odd, even, neither odd nor even, both odd and even)

Answer: Since $\cos x, |x|$ and 5 are all even functions (each graph is symmetric with respect to y-axis), the sum is also even. In other words, $g(-x) = g(x)$ and thus g is an even function.

(b) Find all x satisfying $|4x - 8| < 6$. You may express your answer in interval notation or using inequalities.

Answer: $(1/2, 7/2)$ or $1/2 < x < 7/2$.

We have $|4x - 8| < 6$ is equivalent to $4|x - 2| < 6$ or $|x - 2| < 3/2$. Interpreting $|x - 2|$ as the distance between x and 2, we want the set of points that are less than 3/2 units from 2. Drawing a number line, this gives $1/2 < x < 7/2$.

Alternatively, we may solve the inequality $-6 < 4x - 8 < 6$ by adding 8 and dividing by 4 on both sides.

(c) Simplify $\log_2(1/32) + \ln(e^{12})$.

Answer: 7

Since $2^{-5} = 1/2^5 = 1/32$, we have $\log_2(1/32) = -5$. Then, because $\ln x$ and e^x are inverses, $\ln(e^{12}) = 12$.

(d) Find the equation of the line passing through the point $(1, -6)$ and perpendicular to the line $5x - 10y = 2016$. Give your answer in slope-intercept form.

Answer: $y = -2x - 4$.

First, we compute the slope of the line $5x - 10y = 2016$ by writing it in slope-intercept form: $y = \frac{1}{2}x - 2016/10$. Thus, $m = 1/2$. The slope of a line perpendicular to this has slope $m_\perp = -2$. Therefore, our line has the form $y = -2x + b$. To find b, we substitute $x = 1$ and $y = -6$ into the previous equation to obtain $-6 = -2 \cdot 1 + b$, which gives $b = -4$.

(e) Find the inverse of the function $f(x) = \ln(5 + 2x)$.

Answer: $f^{-1}(x) = \dfrac{e^x - 5}{2}$

Beginning with $y = \ln(5 + 2x)$, we interchange the variables $x = \ln(5 + 2y)$ and solve for y. To isolate y we invert the ln function by raising both sides to the base e.

$$e^x = e^{\ln(5+2y)} = 5 + 2y.$$

Then, subtracting 5 and dividing by 2 on both sides yields the solution.

(f) Complete the square to find the minimum value of the function $Q(x) = 3x^2 + 12x - 1$.

Answer: -13

To complete the square, we first factor out a 3, leaving the -1 outside the parentheses.

$$Q(x) = 3(x^2 + 4x + \underline{\qquad}) - 1 + \underline{\qquad}.$$

Next, we determine the constant to add inside the parentheses by taking half of 4 and squaring. This yields 4. We add 4 inside the parentheses which means that we are really adding $3 \cdot 4 = 12$ to the function. To balance this out, we subtract 12 outside the parentheses:

$$Q(x) = 3\left(x^2 + 4x + 4\right) - 1 - 12 = 3(x + 2)^2 - 13.$$

Since the graph of $Q(x)$ is a parabola opening up, the minimum value of the function is the y-coordinate of the vertex of the parabola. The vertex is $(-2, -13)$ so the minimum value is -13. Another way to see this is to observe that the term in the parentheses is zero only when $x = -2$. Thus, the minimum of the function occurs at $x = -2$ and is found as $Q(-2) = -13$.