KEY

# MATH 135 Calculus 1, Spring 2016

## Worksheet for Sections 1.4 and 1.5

## 1.4 Trigonometric Functions

### Radians

Angles in trigonometry are measured in radians, which corresponds to an actual physical length (as opposed to degrees, which is based on the fact that 360 has many factors).

**Definition 0.1** *An angle of 1 radian is equal to the angle made by 1 unit of arc length along the unit circle.*

To convert between radians and degrees, use the formula

$$\boxed{180^\circ \;=\; \pi \text{ radians.}} \tag{1}$$

**Exercise 0.2** *Convert each of the following from radians to degrees or vice-versa.*

a. $270^\circ = \underline{\frac{3\pi}{2}}$ b. $-135^\circ = \underline{\frac{-3\pi}{4} \text{ or } \frac{5\pi}{4}}$ c. $5\pi/6 = \underline{150^\circ}$ d. $-4\pi/3 = \underline{-240^\circ}$

### The Sine and Cosine Functions

There are multiple definitions of the sine function $y = \sin\theta$, but one of the simplest to understand is that it represents the $y$-coordinate on the unit circle at an angle of $\theta$ radians.

**Definition 0.3** *The sine of $\theta$, denoted by $\sin(\theta)$ or just $\sin\theta$, is the $y$-coordinate of the point of intersection between the unit circle and a ray emanating from the origin at an angle of $\theta$ radians. The cosine of $\theta$, denoted by $\cos(\theta)$ or just $\cos\theta$, is the $x$-coordinate.*

It is important to remember that the input into the functions $f(\theta) = \sin\theta$ or $g(\theta) = \cos\theta$ is an *angle* $\theta$. Since the unit circle has a radius of one, the $x$- and $y$-coordinates of any point on the unit circle always lie between $-1$ and 1. Thus, the range of $\sin\theta$ and $\cos\theta$ is $[-1, 1]$. By the Pythagorean Theorem, we have a fundamental relationship between $\sin\theta$ and $\cos\theta$:

$$\boxed{\cos^2(\theta) + \sin^2(\theta) \;=\; 1 \quad \text{for any angle } \theta.} \tag{2}$$

Another definition for the sine or cosine of an angle comes from using right triangles. This only works for finding the trig function of an angle between 0 and $\pi/2$. Recall the mnemonic phrase SOH-CAH-TOA, which reminds us of the following definitions:

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}}, \qquad \cos\theta = \frac{\text{adj.}}{\text{hyp.}}, \qquad \tan\theta = \frac{\text{opp.}}{\text{adj.}}.$$

**Exercise 0.4** *Without using a calculator, evaluate each of the following:*

a. $\sin(3\pi/2) = \underline{-1}$ b. $\cos(45\pi) = \underline{-1}$ c. $\sin(7\pi/6) = \underline{-1/2}$ d. $\tan(3\pi/4) = \underline{-1}$

**Other Trig Functions**

There are four other trigonometric functions, each of which can be defined in terms of $\sin\theta$ and $\cos\theta$.

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}.} \tag{3}$$

**Exercise 0.5** *Use the fundamental identity given in equation (2) to derive the identity*

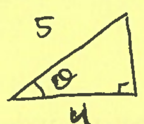
$$\tan^2\theta + 1 = \sec^2\theta.$$

start with $\cos^2\theta + \sin^2\theta = 1$ and divide both sides by $\cos^2\theta$.

$\Rightarrow \dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$ or $1 + \tan^2\theta = \sec^2\theta$ since $\tan^2\theta = \left(\dfrac{\sin\theta}{\cos\theta}\right)^2 = \dfrac{\sin^2\theta}{\cos^2\theta}$. QED.

**Exercise 0.6** *Suppose that* $\sin\theta = \dfrac{3}{5}$ *and that* $\pi/2 < \theta < \pi$. *Find the values of* $\cos\theta, \tan\theta$, *and* $\csc\theta$.

use SOH-CAH-TOA or $\cos^2\theta + \sin^2\theta = 1$.




$\therefore \cos\theta = \frac{4}{5}$ except that $\pi/2 < \theta < \pi \Rightarrow \theta$ in second quadrant.

3, 4, 5 right triangle.

Thus, $\cos\theta = -4/5$ (x-coord. is neg.)

$\tan\theta = \dfrac{3/5}{-4/5} = -3/4$ and $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{5}{3}$

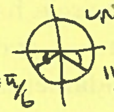
**Exercise 0.7** *Find all solutions to the equation* $\sin\beta = -\dfrac{1}{2}$ *assuming that* $0 \le \beta \le 2\pi$.

we want to find the angle $\beta$ between $0$ and $2\pi$ whose sine is $-1/2$.

First, find the reference angle that has a sine of $1/2$.


$\Rightarrow \sin 30^\circ = 1/2$ so $\sin \pi/6 = 1/2$.

unit circle




$\dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

**Exercise 0.8** *Sketch a graph of the following trig functions. State the amplitude and period in each case.*

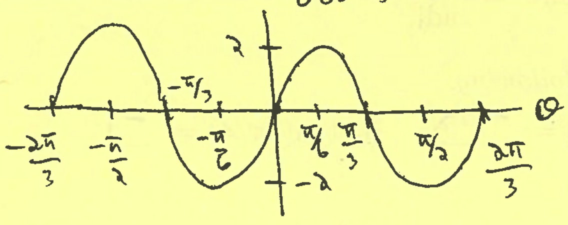
**a.** $y = 2\sin(3\theta)$

period: $\dfrac{2\pi}{|b|}$

amplitude: 2

period: $\dfrac{2\pi}{3}$

odd function



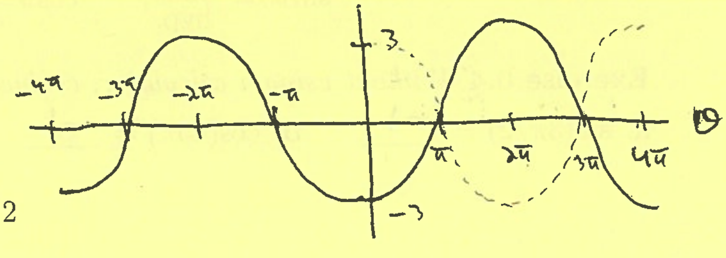

**b.** $y = -3\cos\left(\dfrac{\theta}{2}\right)$

− sign means reflect about θ-axis

amplitude: 3

period: 4π

even function



## 1.5 Inverse Functions

One of the simplest ways to obtain a new function from an old one is to simply flip the domain and range. So if $f(2) = 7$, then the new function $f^{-1}$, called the **inverse of** $f$, has $f^{-1}(7) = 2$. The function $f^{-1}$ simply maps each element in the range of $f$ back to the element in the domain it came from; it "inverts" $f$.

However, there will be a problem if an element $b$ in the range has two or more elements in the domain, say $a_1$ and $a_2$, that map to it (i.e., $f(a_1) = f(a_2) = b$). Then, the inverse of $b$ would not be unique (it could be either $a_1$ or $a_2$), and $f^{-1}$ would not be a function. To rectify this, we must assume that $f$ is **one-to-one**, that is, each element in the range of $f$ has one and only one pre-image that was sent to it in the domain.

For example, the function $f(x) = x^2$ is not one-to-one because both $-2$ and $2$ are each sent to the same element in the range, $f(-2) = f(2) = 4$. This function fails the **horizontal line test** and is really two-to-one. However, if we restrict the domain to $x \geq 0$ (think of erasing the left half of the parabola), then $f$ becomes one-to-one and now the inverse is actually a function. You know it already as $f^{-1}(x) = \sqrt{x}$. By definition (i.e., restricting the domain of $x^2$), $\sqrt{x}$ only spits out non-negative values.

**Key Point:** The only functions with well-defined inverses are those that are **one-to-one**. They must pass the **horizontal line test.**

Note that the notation for the inverse of $f$ is not the usual exponent notation. In other words,

$$f^{-1} \neq \frac{1}{f}$$

The choice of $-1$ as the exponent is mathematical shorthand for **inverse**. Don't confuse this!

Based on the definition of the inverse of a function, the following formulas should make sense:

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x.$$

Simply put, the inverse of $f$ reverses what $f$ does to $x$. Likewise, $f$ reverses the action of $f^{-1}$, that is, the inverse of $f^{-1}$ is just $f$.

Graphically, if $(x, y)$ is a point on the graph of $f$, then $(y, x)$ is a point on the graph of $f^{-1}$. Thus, to obtain the graph of $f^{-1}$ from the graph of $f$ (or vice versa), just reflect the graph of $f$ about the line $y = x$. This is also how one obtains an analytic formula for the inverse of a function: interchange the variables $x$ and $y$ and solve for $y$.

**Exercise 0.9** *Find the inverse of the function $f(x) = 1 + \sqrt{2 + 3x}$. What should the domain of $f^{-1}$ be in order to have $f$ as its inverse?*

$y = 1 + \sqrt{2+3x}$

Domain $f$: $x \geq -2/3 \Rightarrow$ Range $f^{-1}$ is $[-\frac{2}{3}, \infty$

Range $f$: $y \geq 1 \Rightarrow$ Domain $f^{-1}$ is $[1, \infty)$

Switch variables:

$x = 1 + \sqrt{2+3y}$

Solve for $y$:

$x - 1 = \sqrt{2+3y}$

$(x-1)^2 = 2 + 3y$

$(x-1)^2 - 2 = 3y \Rightarrow$

$y = \frac{1}{3}(x-1)^2 - \frac{2}{3}$

½ a parabola

$f^{-1}(x) = \frac{1}{3}(x-1)^2 - \frac{2}{3}$

over the domain $[1, \infty)$

## Inverse Trig Functions

The key to defining the inverse trig functions is to restrict the domains of the original trig functions in order to ensure that they are one-to-one. For example, the sine function is one-to-one on the domain $-\pi/2 \leq \theta \leq \pi/2$ (check the graph). By making this restriction, we then **define** the range of the inverse sine function (also called the **arcsine function**) to be $[-\pi/2, \pi/2]$. The domain of the inverse sine function is $[-1, 1]$ because this is precisely the range of the sine function.

**Key Point:** The inverse sine function, denoted $\sin^{-1}(x)$, inputs numbers between $-1$ and 1 and outputs angles between $-\pi/2$ and $\pi/2$. If $\theta = \sin^{-1}(x)$, then $\sin(\theta) = x$. (Go backwards!)

The outputs of the other inverse trig functions are given below:

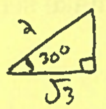
- Range of $\cos^{-1}(x)$ is $[0, \pi]$.
- Range of $\sec^{-1}(x)$ is $[0, \pi]$ (except for $\pi/2$).
- Range of $\tan^{-1}(x)$ is $(-\pi/2, \pi/2)$.
- Range of $\cot^{-1}(x)$ is $(0, \pi)$.
- Range of $\csc^{-1}(x)$ is $[-\pi/2, \pi/2]$ (except for 0).

**Exercise 0.10** *Evaluate each of the following without using a calculator:*

**a.** $\sin^{-1}(1/\sqrt{2}) =$ $\pi/4$ **b.** $\sin^{-1}(1) =$ $\pi/2$ **c.** $\cos^{-1}(-\sqrt{3}/2) =$ $5\pi/6$

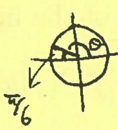
**d.** $\tan^{-1}(-1) =$ $-\pi/4$ **e.** $\sec^{-1}(2) =$ $\pi/3$ **f.** $\csc^{-1}(-1) =$ $-\pi/2$

c. $\cos^{-1}(-\sqrt{3}/2)$ means find the angle $\theta$ so that $\cos\theta = -\sqrt{3}/2$. (backwards)




$\cos(30°) = \frac{\sqrt{3}}{2}$ or $\cos(\pi/6) = \sqrt{3}/2$

ref. angle



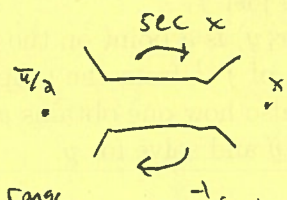

$\theta = \pi - \pi/6$

$= 5\pi/6$.

**Exercise 0.11** *Explain why* $\pi/2$ *must be excluded from the range of* $\sec^{-1}(x)$. *In other words, why does the equation* $\sec^{-1}(x) = \pi/2$ *have no solution?*

b/c $\sec(\pi/2)$ does not exist.

$\sec(\pi/2) = \frac{1}{\cos(\pi/2)} = \frac{1}{0}$ D.N.E.




No possible x works b/c $\sec(\pi/2)$ D.N.E.

**Exercise 0.12** *Explain why* $\cos^{-1}(\cos(17\pi)) = \pi$ *and not* $17\pi$.

It is tempting to cancel out $\cos^{-1}$ and $\cos$. In general, it is true that $f^{-1}(f(x)) = x$. However, $17\pi$ is not in the range of $\cos^{-1}x$. The output of $\cos^{-1}(\ )$ is always between 0 and $\pi$.

So $\cos^{-1}(\cos(17\pi)) = \cos^{-1}(-1) = \pi$ b/c $\cos\pi = -1$ and $\pi$ is in the range of $\cos^{-1}x$.