

MATH 135 Calculus 1, Spring 2016

Worksheet for Sections 2.1 and 2.2

2.1 Limits, Rates of Change, and Tangent Lines

Velocity

Recall that Sir Isaac Newton, one of the co-founders of Calculus, basically invented the subject to verify Kepler's laws of planetary motion. In order to begin, Newton needed to understand the mechanics of motion, in particular, the **velocity** of an object (in his case it was a celestial body such as a planet or the Sun). This section is concerned with distinguishing between the **average velocity** and **instantaneous velocity** of an object. The ideas apply to other functions that change (e.g., population growth, temperature, or the value of a stock), but we will focus on velocity because it is one of the most familiar **rates of change**.

It is traditional to use the variable s to represent the position of an object and use t for time. Suppose we have a function $s(t)$ that describes the position of an object at time t . The input variable is time and the output variable provides the position. The average velocity of the object is given by the ratio of the change in position to the change in time. Recall that Δ = "Delta" means change.

$$\text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{\text{change in position}}{\text{change in time}}.$$

You should be familiar with velocity from traveling in a car. An average velocity of 70 miles/hour means you travel approximately 70 miles every hour. In terms of $s(t)$, the average velocity over the interval between two different times t_1 and t_2 is given by

$$\text{average velocity over } t_1 \leq t \leq t_2 = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \text{slope of secant line.}$$

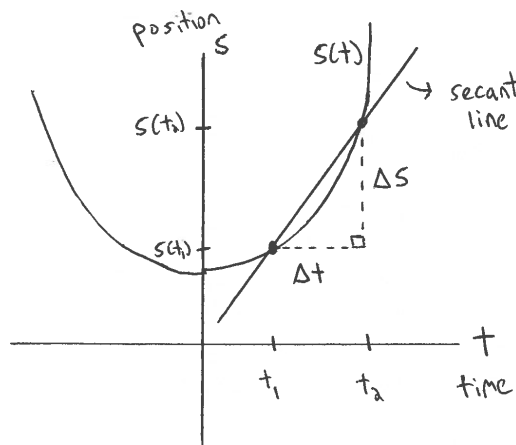


Figure 1: The average velocity over the interval $[t_1, t_2]$ is equal to the slope of the line between the two points on the graph of $s(t)$ at $t = t_1$ and $t = t_2$. This line is called the **secant line**.

To find the instantaneous velocity at time $t = t_1$, we compute the average velocity over smaller and smaller time intervals around t_1 . In theory, the values of the average velocities will begin to **limit** on a specific value, and this will be called the instantaneous velocity. Graphically, this is best visualized by sliding the point at $t = t_2$ down the curve $s(t)$ toward the point at $t = t_1$. As we slide, we compute the slope of the secant line at each step. The limiting value of these slopes (assuming this limit exists) is defined to be the slope of the **tangent line**. This is the fundamental idea behind all of Calculus 1:

$$\begin{aligned} \text{average velocity} &= \text{slope of secant line} = m_{\text{sec}} \\ \text{instantaneous velocity} &= \text{slope of tangent line} = m_{\text{tan}}, \end{aligned}$$

where

$$m_{\text{tan}} = \lim_{t_2 \rightarrow t_1} m_{\text{sec}}.$$

The notation “lim” stands for limit and $t_2 \rightarrow t_1$ means to evaluate the given quantity (in this case, the slope of the secant line) repeatedly as t_2 approaches t_1 . If the limit exists, then the values will approach some specific number, and this number is known as the **limit**. Limits are at the heart of calculus and are the main focus of Chapter 2.

Exercise 0.1 Suppose that $s(t) = t^2 + 3$ describes the position of an object (in feet) at time t (in seconds). Compute the average velocity over each of the following time intervals:

(a) $[1, 2]$

(b) $[1, 1.1]$

(c) $[0.99, 1]$

(d) $[0.999, 1]$

(e) Based on the results above, what is the instantaneous velocity at time $t = 1$? In other words, find the value of $\lim_{t \rightarrow 1} \frac{s(t) - s(1)}{t - 1}$.

2.2 Limits: A Numerical and Graphical Approach

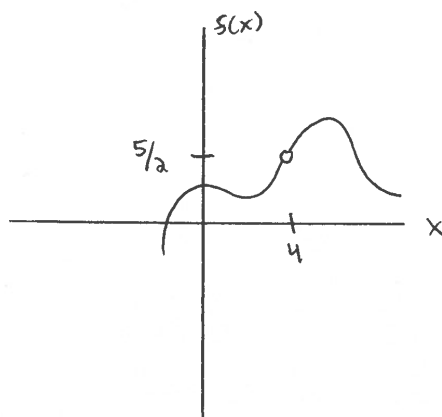
For the previous problem, we computed that

$$\lim_{t \rightarrow 1} \frac{t^2 - 1}{t - 1} = 2.$$

If we define the function $f(t) = \frac{t^2 - 1}{t - 1}$, then we have $\lim_{t \rightarrow 1} f(t) = 2$. Mathematically speaking, this is read, “As t approaches 1, the limit of $f(t)$ approaches 2.” Simply put, as t (the input) gets really, really close to 1, $f(t)$ (the output) gets really, really close to 2.

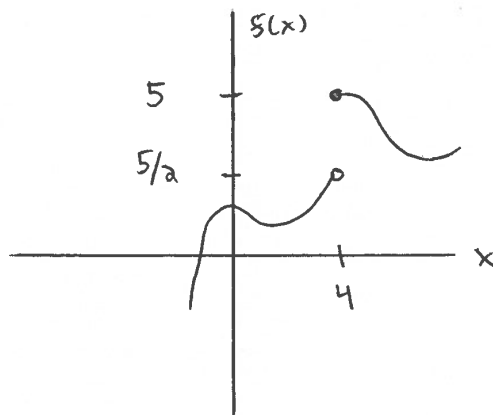
It is important to note that for this example, although $f(1)$ is undefined (can't divide by 0), the limit still exists. When computing the limit of a function, we only care about the value that the function **approaches** as we get close to the point in question. The actual value of the function (or lack thereof) at the destination point is irrelevant.

For example, for the function $f(x)$ graphed below, we have $\lim_{x \rightarrow 4} f(x) = 5/2$, even though $f(4)$ does not exist. Thus, even if the graph has a hole at the destination point ($x = 4$), the limit can still exist.



Left- and right-hand limits

Another important fact about limits is that they must be approaching the same value from **both** directions. For the graph below, $\lim_{x \rightarrow 4} f(x)$ does not exist because the **left-hand limit** at $x = 4$ equals $5/2$, while the **right-hand limit** equals 5. The left-hand limit is computed by only considering the function values for $x < 4$, while the right-hand limit is computed by restricting to function values for $x > 4$. These values must be the same for the limit to exist. The notation is shown below.



Left-hand limit: $\lim_{x \rightarrow 4^-} f(x) = 5/2$

Right-hand limit: $\lim_{x \rightarrow 4^+} f(x) = 5$

Key Fact: $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$.

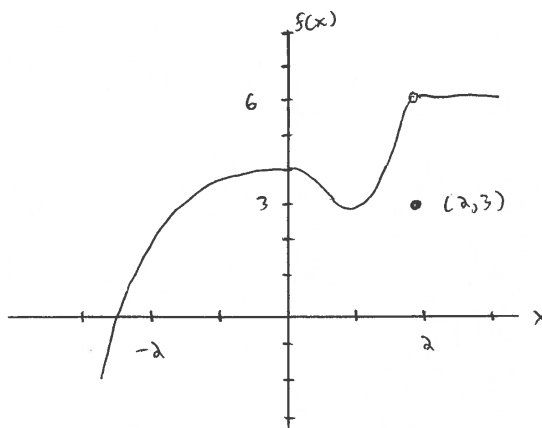
Exercise 0.2 Evaluate each of the following limits using the graph of $f(x)$ shown below.

(a) $\lim_{x \rightarrow 2^-} f(x)$

(b) $\lim_{x \rightarrow 2^+} f(x)$

(c) $\lim_{x \rightarrow 2} f(x)$

(d) Find $f(2)$.



Exercise 0.3 Find each of the following limits. Note that ∞ or $-\infty$ are acceptable answers.

(a) $\lim_{x \rightarrow 3^-} \frac{1}{x-3}$

(b) $\lim_{x \rightarrow 3^+} \frac{1}{x-3}$

(c) $\lim_{x \rightarrow 3} \frac{1}{x-3}$

(d) $\lim_{x \rightarrow 0^+} \ln x$

Exercise 0.4 Use a calculator to find the value of $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$. Does your answer depend on whether your calculator is in degrees or radians?

Exercise 0.5 Use a calculator to find the value of $\lim_{x \rightarrow 0} \frac{2^x - 1}{x}$. Give your answer to five decimal places.

Exercise 0.6 What is the difference between each of the following limits? Explain.

(a) $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$

(b) $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$