KEY

MATH 135 Calculus 1, Fall 2015

Worksheet on Applied Optimization (Section 4.7)

Some comments:

1. It's all about the set up! Draw a picture and label variables. The eventual goal is to arrive at a **function of one variable** representing a quantity to be optimized. For example, if you are finding the smallest (minimal) surface area S, then you want to find an equation for S as a function of one variable. So a formula like $S = 2w^2 + 4wl$ needs to be reduced to a formula with just w or l on the right-hand side. Usually there will be some condition given that will allow you to substitute in for one variable to accomplish this reduction. Once you have a function of one variable, find the max or min by the techniques we have been discussing in class (e.g., take the derivative and set it equal to 0 to find the critical points). Don't forget to check that your solution really is a max or a min.

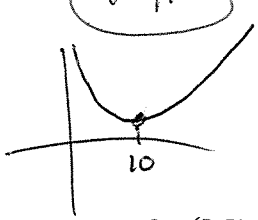
2. Word problems are hard. They are hard for everyone — students, grad students, even professors. It's fine to get discouraged or frustrated. This is to be expected. But they are also really important! Remember that calculus is essentially an applied subject and that problem solving is what people do in the "real world." No one is going to offer you a job because you can take the derivative of a function. People like to hire good problem solvers and that's where getting proficient at doing word problems really pays off (pun intended).

Problems

1. **(Warm-Up)** Find two positive numbers whose product is 100 and whose sum is a minimum.

   Ans: Start by calling the two numbers x and y. You want to minimize the quantity $S = x + y$. Before you can do this you need to write S as a function of one variable. Find a relationship between x and y and then use this to substitute into the right-hand side of S to get a function of one variable. Then find the minimum of this function and solve the problem. Be sure to check that your solution really is an absolute minimum.

10, 10



$xy = 100$

$S = x + y = x + \frac{100}{x}$

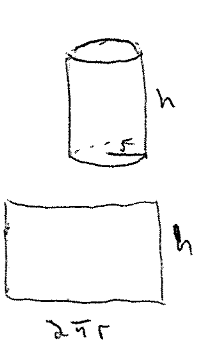
$S'(x) = 1 - \frac{100}{x^2} = 0 \Rightarrow x = 10$

$S'' = \frac{200}{x^3} > 0$ for $x > 0$

$\Rightarrow$ concave up

$\Rightarrow$ $x = 10$ is a min.

2. **(Minimizing Surface Area)** An aluminum can needs to be designed to hold 100 cm^3 of juice. The can is cylindrical with flat caps at both ends. Find the dimensions of the can that use the least amount of material.



$V = \pi r^2 h = 100 \Rightarrow h = \frac{100}{\pi r^2}$

$S = 2\pi r^2 + 2\pi r h$

$S = 2\pi r^2 + 2\pi r \cdot \frac{100}{\pi r^2}$

$S = 2\pi r^2 + \frac{200}{r}$

$S' = 4\pi r - \frac{200}{r^2} \Rightarrow r^3 = \frac{50}{\pi}$

$S'' = 4\pi + \frac{400}{r^3} > 0$ (min)

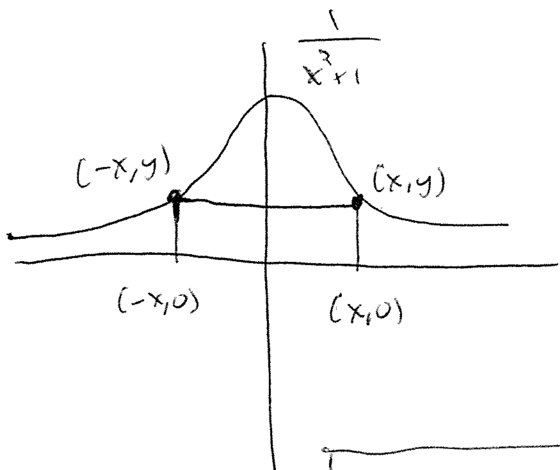
$r = \sqrt[3]{\frac{50}{\pi}} \approx 2.5154$

$h = \frac{100}{\pi \left(\frac{50}{\pi}\right)^{2/3}} = \frac{2 \cdot 50}{50^{2/3} \cdot \pi^{1/3}}$

$h = 2\sqrt[3]{\frac{50}{\pi}} \approx 5.0308$

3. **(Maximizing Area)** A rectangle has one side on the x-axis and two vertices on the curve $y = \dfrac{1}{1+x^2}$. Find the vertices of the rectangle with maximum area.

Ans: First, use your curve sketching techniques to sketch the graph of the function. Notice that it is an even function. Then draw a rectangle with the base on the x-axis whose upper vertices are on the curve. What symmetry do you notice about your rectangle? Label the lower right vertex $(x, 0)$ and find the area $A(x)$ of the rectangle as a function of x. Find where A has a maximum and finish the problem.



$y = \frac{1}{1+x^2}$ $\quad A = 2x \cdot y = \frac{2x}{1+x^2}$

$A'(x) = \frac{2(1-x^2)}{(1+x^2)^2} \Rightarrow x = 1$

A' $\quad + \quad 0 \quad -$ $\quad$ (max) at 1

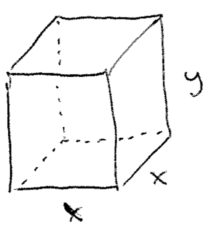
$A(0) = 0$

$A(\infty) = 0$

no other c.p.'s for $x > 0$.

$\boxed{(1,0),\ (-1,0),\ (1, \tfrac{1}{2}),\ (-1, \tfrac{1}{2})}$

4. **(Minimizing Cost)** A jewelry box with a square base is to be built with copper plated sides and a nickel plated bottom and top. The volume of the box is 32 cm^3. If nickel plating costs \$3 per cm^2 and copper plating costs \$6 per cm^2, find the dimensions of the box to minimize the total cost of the materials.



$V = x^2 y = 32$

$C = 3 \cdot x^2 + 3 \cdot x^2 + 6 \cdot xy \cdot 4 = 6x^2 + 24xy$

$V = 32 \Rightarrow y = \frac{32}{x^2} \Rightarrow C = 6x^2 + \frac{24x \cdot 32}{x^2}$

$= 6x^2 + \frac{24 \cdot 32}{x}$

$C'(x) = 12x - \frac{24 \cdot 32}{x^2} = 0 \Rightarrow \cancel{12}x = \frac{\overset{2}{\cancel{24}} \cdot 32}{x^2} \Rightarrow x^3 = 64$

$\Rightarrow x = 4$

$y = \frac{32}{16} = 2$

$C''(x) = 12 + 2 \cdot 24 \cdot 32\, x^{-3} > 0$

concave up $\Rightarrow$ c.p. is a min. ✓

$\boxed{\text{base } 4 \text{ cm, height } 2 \text{ cm}}$