MATH 135 Calculus 1, Spring 2016

Worksheet for Sections 2.6 and 2.7

2.6 Trigonometric Limits

This section focuses on two key limits involving $\sin x$ and $\cos x$ that are important for finding the slope of the tangent line to each function. These limits are proven through an important and intuitive theorem called the **Squeeze Theorem** (discussed previously in Section 2.3).

Theorem 0.1 (The Squeeze Theorem) *Suppose that $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at $x = a$) and that*

$$\lim_{x \to a} f(x) \;=\; \lim_{x \to a} h(x) \;=\; L.$$

Then $\lim_{x \to a} g(x) = L$.

The Squeeze Theorem states that if one function is "squeezed" between two others having a common limit, then the inner function takes on the same limit. It is best understood visually (see Figure 2 on p. 89 of the text).

Exercise 0.2 *Suppose that $g(x)$ satisfies $\cos x \leq g(x) \leq x^2 + 1$ for x-values near $x = 0$. Use the Squeeze Theorem to find $\lim_{x \to 0} g(x)$.*

Two important trigonometric limits are

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} \;=\; 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos x}{x} \;=\; 0.} \tag{1}$$

These can be checked by using a calculator (set it to radians!) and plugging in values very close to 0. Note that each limit takes the form of $\frac{0}{0}$, an indeterminate form.

To prove the first limit in Equation (1), we use the fact that (see Figure 1)

$$\cos x \;\leq\; \frac{\sin x}{x} \;\leq\; 1 \qquad \text{for } -\pi/2 < x < \pi/2.$$

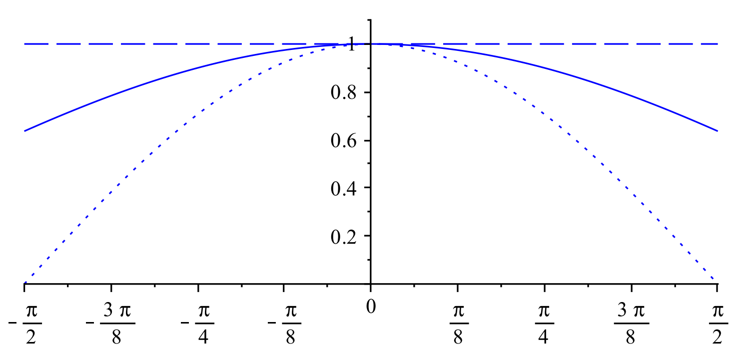

Figure 1: The graphs of the functions $y = 1$ (dashed), $y = \sin(x)/x$ (solid), and $y = \cos x$ (dotted).

Since $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} 1 = 1$, by the Squeeze Theorem, we have that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$, as desired.

Exercise 0.3 *Using the limit* $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$*, show that* $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$.

Hint: *Multiply the top and bottom of* $\frac{1 - \cos \theta}{\theta}$ *by* $1 + \cos \theta$*, simplify, and break the fraction into the product of two fractions, one of which is* $\frac{\sin \theta}{\theta}$*. Then use the fact that the limit of a product is the product of the limits.*

Exercise 0.4 *Evaluate* $\lim_{x \to 0} \frac{\sin^2 x}{x^2}$. **Hint:** *The limit of the product equals the product of the limits.*

Exercise 0.5 *Use a calculator to evaluate* $\lim_{t \to 0} \frac{\sin(7t)}{t}$*. Then verify your answer by making the substitution* $x = 7t$.

Hint: *If* t *is tending toward 0, and* $x = 7t$*, then what is* x *approaching? Try and rewrite the limit using only the variable* x *so that the fraction* $\frac{\sin x}{x}$ *is present.*

Exercise 0.6 *Evaluate* $\lim_{\theta \to 0} \frac{\sin(9\theta)}{5\theta}$.

2.7 Limits at Infinity

The expression $\lim_{x\to\infty} f(x)$ means to calculate the function values of f as x gets larger and larger, and see if they approach a limit. As with usual limits, the answer may be a real number L, ∞, $-\infty$, or the limit may not exist. For example,

$$\lim_{x\to\infty} x^2 = \infty$$

because as x gets larger, the value of x^2 gets even larger, and is therefore going to ∞. On the other hand, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

because as x gets larger, $1/x$ gets smaller and smaller. We say that the function $f(x) = 1/x$ has a **horizontal asymptote** at $y = 0$ because the graph of f approaches the horizontal line $y = 0$ as x tends to ∞.

The expression $\lim_{x\to-\infty} f(x)$ means to calculate the function values of f as x gets larger and larger, but negative. For example, we have

$$\lim_{x\to-\infty} x^2 = \infty, \qquad \lim_{x\to-\infty} x^3 = -\infty, \qquad \text{and} \qquad \lim_{x\to-\infty} e^x = 0.$$

Exercise 0.7 *Evaluate each of the following limits, if they exist.*

a. $\lim_{x\to\infty} e^{2x}$

b. $\lim_{x\to\infty} e^{-2x}$

c. $\lim_{x\to\infty} -x^4 + 3x^2 + 7$

d. $\lim_{x\to\infty} \sin x$

e. $\lim_{x\to-\infty} 2x^2 - 3x^3$

f. $\lim_{x\to\infty} \tan^{-1} x$

g. $\lim_{x\to-\infty} \tan^{-1} x$

Limits of Rational Functions

Consider the following limit:

$$\lim_{x\to\infty} \frac{3x^2 + 5x - 2}{4x^2 + 7}$$

To find it, we divide the top and bottom of the fraction by the **highest power in the denominator**, which in this case is x^2. This gives

$$\lim_{x\to\infty} \frac{3x^2 + 5x - 2}{4x^2 + 7} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} + \frac{5x}{x^2} - \frac{2}{x^2}}{\frac{4x^2}{x^2} + \frac{7}{x^2}} = \lim_{x\to\infty} \frac{3 + \frac{5}{x} - \frac{2}{x^2}}{4 + \frac{7}{x^2}} = \frac{3}{4},$$

since each remaining fraction in the numerator and denominator is heading to 0 as x tends to ∞.

Exercise 0.8 *Evaluate* $\lim_{x\to\infty} \frac{6x^4 - 5x^3 + 2x}{4x^2 - 7x^4 + 1}$.

Exercise 0.9 *Evaluate* $\lim_{x\to\infty} \frac{6x^4 - 5x^3 + 2x}{4x^5 - 7x^4 + 1}$.

Exercise 0.10 *Evaluate* $\lim_{x\to\infty} \frac{\sqrt{16x^4 + 10}}{3x^2 + 1}$. **Hint:** *Ignore the* 10 *in the numerator. (Why is it ok to do this?)*

Exercise 0.11 *Evaluate* $\lim_{x\to\infty} \frac{e^x + 3e^{-x}}{2e^x - e^{-x}}$. **Hint:** *What is the "highest" power in the denominator? Try dividing top and bottom of the fraction by it.*

Exercise 0.12 *Evaluate* $\lim_{x\to\infty} \tan^{-1}\left(\frac{5x^3 - 2x^2 + 3}{5x^3 + 9x^2 - 3x + \pi}\right)$.