MATH 135 Calculus 1, Spring 2016

Worksheet for Sections 3.3 and 3.5

3.3 Product and Quotient Rules

There are two useful rules for computing the derivative of a product and quotient of two functions. Interestingly, Leibniz himself messed up the product rule in an early draft of his manuscript on calculus.

Theorem 0.1 (Product Rule) *If $f(x)$ and $g(x)$ are differentiable functions, then so is their product $f(x) \cdot g(x)$. The derivative of the product is given by*

$$\boxed{\frac{d}{dx}\left(f(x) \cdot g(x)\right) \;=\; f'(x)g(x) + f(x)g'(x)\,.} \tag{1}$$

Notice the symmetry in Formula (1) and that it is **not** the case that the derivative of the product equals the product of the derivatives. For instance, suppose we applied the Product Rule to take the derivative of $x \cdot x$. If we just multiplied the product of the derivatives, we would get

$$\frac{d}{dx}(x \cdot x) \;=\; \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) \;=\; 1 \cdot 1 \;=\; 1\,???$$

But this is clearly incorrect since $x \cdot x = x^2$ and the derivative of x^2 is $2x$ by the Power Rule. A correct application of the Product Rule is as follows:

$$\frac{d}{dx}(x \cdot x) \;=\; \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) \;=\; 1 \cdot x + x \cdot 1 \;=\; 2x.$$

Exercise 0.2 *Use the Product Rule to find $f'(x)$ where $f(x) = (3x^2+1)e^x$. Simplify your answer.*

Theorem 0.3 (Quotient Rule) *If $f(x)$ and $g(x)$ are differentiable functions, then so is their quotient $f(x)/g(x)$ as long as $g(x) \neq 0$. The derivative of the quotient is given by*

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \;=\; \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,.} \tag{2}$$

Note: The Quotient Rule can be derived from the Product Rule. Start by letting $Q(x) = \dfrac{f(x)}{g(x)}$. Then cross multiply and differentiate both sides with respect to x using the Product Rule. Solving for $Q'(x)$ leads to formula (2).

Exercise 0.4 *Use the Quotient Rule to calculate the derivative of* $\frac{1}{x^4}$ *and check your answer against the result obtained from the Power Rule.*

Exercise 0.5 *If* $g(x) = \frac{3x+1}{2x-5}$, *find and simplify* $g'(x)$.

Exercise 0.6 *If* $h(x) = \frac{e^x}{x^2+1}$, *find and simplify* $h'(x)$.

3.5 Higher Derivatives

Recall that the derivative $f'(x)$ is the **slope function**: it gives the slope of the function f at the point x. Since $f'(x)$ is itself a function, we can ask, "What is the derivative of the derivative?" This is called the **second derivative.**

Definition 0.7 $\frac{d}{dx}(f'(x)) = f''(x)$ *is called the* <u>*second derivative*</u>. *It measures the rate of change of the slope function* $f'(x)$.

To find the second derivative of a function f, we simply take the derivative twice. For example, if $f(x) = 3x^2 - 5x$, then $f'(x) = 6x - 5$ and $f''(x) = 6$, applying the Power Rule twice. Note that in this particular case, f is a quadratic function, f' is a linear function and f'' is a constant function. Each time we take the derivative, the power decreases by one.

Exercise 0.8 *Suppose that* $f(x) = 2x^4 - 3x + e^x$. *Find* $f'(x)$ *and* $f''(x)$.

Leibniz Notation for the Second Derivative

To write the second derivative using Leibniz notation, we use

$$f''(x) \;=\; \frac{d^2 f}{dx^2} \quad \text{or} \quad \text{if } y = f(x), \text{ then } f''(x) \;=\; \frac{d^2 y}{dx^2}\,.$$

Concavity

If f' tells us the slope of the function, what does f'' represent? To answer this question, note that if $f'(x) > 0$, then the slope is positive at x and the function is **increasing** there (moving upwards from left to right). On the other hand, if $f'(x) < 0$, then the slope is negative at x and the function is **decreasing** there (moving downwards from left to right).

$$\boxed{\begin{array}{c} f' > 0 \implies f \text{ is increasing} \\ f' < 0 \implies f \text{ is decreasing} \end{array}} \tag{3}$$

Next, suppose that $f''(x) > 0$. This means that $(f')' > 0$, so the slopes of f are increasing. There are two possibilities. If the slopes are positive and getting bigger, then the curve is getting steeper, increasing at a faster and faster rate (like e^x). On the other hand, the slopes could be negative but getting closer to 0 (e.g., $m = -3, m = -1.4, m = -0.4$). In this case, the curve is decreasing but beginning to flatten out (such as e^{-x}). In either case, we say that the graph of f is **concave up**.

Similarly, if $f''(x) < 0$, then the slopes are decreasing. There are two cases here as well. Either the slopes are positive and getting smaller (like $\ln x$) or they are negative and getting more negative (such as $\ln(-x)$ for $x < 0$). In the first case, the curve is increasing, but starting to flatten out, while in the second case, the curve is decreasing and becoming more and more steep. In either of these cases, we say that the graph of f is **concave down**. In sum, we have

$$\boxed{\begin{array}{c} f'' > 0 \implies f \text{ is concave up} \\ f'' < 0 \implies f \text{ is concave down} \end{array}} \tag{4}$$

Exercise 0.9 *Sketch the graph of a function f such that $f'(x) < 0$ and $f''(x) > 0$ everywhere along the function.*

Exercise 0.10 *Former President Nixon once famously said, "Although the rate of inflation is increasing, it is increasing at a decreasing rate." (He was trying to assuage the fears of a nervous US public about the rapidly rising cost of goods.) Let $r(t) =$ the rate of inflation. What are the signs ($+, -$ or 0) of $r'(t)$ and $r''(t)$?*

Higher Derivatives

Just as we can take the derivative of the derivative, we can also take the derivative of the second derivative, called the **third derivative**. Thus, the third derivative of f, denoted as $f'''(x)$, means to take the derivative of f three times. It represents the change in the concavity of a function.

But there is no reason to stop at just three derivatives. The fourth derivative, denoted $f^{iv}(x)$, is the derivative of $f'''(x)$ and represents the concavity of the second derivative or reveals the slopes of f'''. Although this might appear to be a twisted mathematical extension of the derivative, higher derivatives frequently arise in applications. For instance, if $s(t)$ represents the position of a particle, then the third derivative $s'''(t)$ is known as the **jerk** (from physics). The fourth derivative shows up in differential equations that model the flow of waves or turbulence around an airplane.

Exercise 0.11 *Suppose that $f(x) = xe^x$. Find and simplify $f'(x), f''(x)$ and $f'''(x)$. In general, what is $f^{(n)}(x)$, the nth derivative of f. Try and find a formula that involves n.*