Key

MATH 135 Calculus 1, Fall 2015

Worksheet for Sections 4.3, 4.4, and 4.6

Derivatives and Curve Sketching

Key Idea: The material in these three sections focuses on using the first and second derivative of a function to obtain a good sketch of the graph of the function. They also can be used to determine the type of critical point (max, min, or neither).

First, let us review the key information revealed by the signs of the first and second derivatives. The first derivative indicates whether a function is increasing (positive slope), decreasing (negative slope), or neither (critical point; zero slope).

$$\boxed{\begin{array}{c} f'(x) > 0 \text{ for } x \in (a,b) \implies f \text{ is increasing on } (a,b) \\ f'(x) < 0 \text{ for } x \in (a,b) \implies f \text{ is decreasing on } (a,b) \\ f'(c) = 0 \implies c \text{ is a critical point of } f \end{array}}$$

The second derivative indicates whether the function is concave up ($f'' > 0$) or concave down ($f'' < 0$). Concave up means that the first derivative is increasing (slopes increasing) while concave down means that the first derivative is decreasing (slopes decreasing).

$$\boxed{\begin{array}{c} f''(x) > 0 \text{ for } x \in (a,b) \implies f \text{ is concave up on } (a,b) \\ f''(x) < 0 \text{ for } x \in (a,b) \implies f \text{ is concave down on } (a,b) \end{array}}$$

An **inflection point** is a point where the concavity changes. It can be found by solving the equation $f''(x) = 0$ and then checking that the sign of f'' flips to either side of a solution.

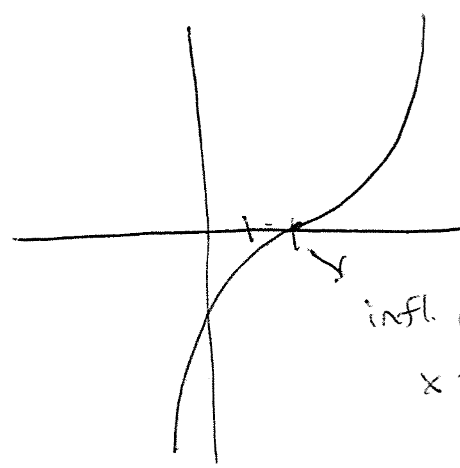
Example 1: The function $f(x) = (x-2)^3$ has one inflection point. Find it. On what interval is f concave up? concave down? Sketch the graph of f.
Hint: Draw a second derivative number line indicating where f'' is positive, negative, or 0.

$f'(x) = 3(x-2)^2 \cdot 1$

$f''(x) = 6(x-2)$

$-$ $\quad$ 0 $\quad$ $+$ $\quad$ f''

2



infl. pt at $x = 2$

Concave down: $(-\infty, 2)$

Concave up: $(2, \infty)$

By checking the sign of the first derivative to either side of a critical point, the first derivative can be used to determine the type of critical point. This is called the **first derivative test**. For instance, if $f'(4) = 0$, then $x = 4$ is a critical point. If $f' > 0$ for x slightly less than 4 and $f' < 0$ for x slightly greater than 4, we know that $x = 4$ is a local maximum (the function increases as x approaches 4, then begins to decrease.)

First Derivative Test: Suppose that $x = c$ is a critical point of f.

$f'(x)$ changes from $+$ to $-$ at c	$\Longrightarrow$	c is a local max
$f'(x)$ changes from $-$ to $+$ at c	$\Longrightarrow$	c is a local min

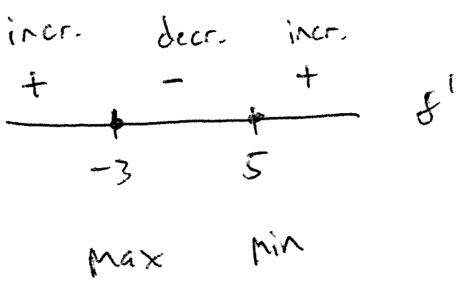
Note: If f' does not change sign at a critical point, then $x = c$ is neither a max nor a min.

Example 2: Consider the function $f(x) = x^3 - 3x^2 - 45x + 5$. Find the critical points and use the first derivative test to classify each critical point as a local max, local min, or neither. On what interval(s) is f increasing? decreasing? Sketch a graph of f.
Hint: Draw a first derivative number line indicating where f' is positive, negative, or 0.

$$f'(x) = 3x^2 - 6x - 45$$
$$= 3(x^2 - 2x - 15)$$
$$= 3(x-5)(x+3)$$

$x = 5$ $\quad$ $x = -3$

crit. pts.

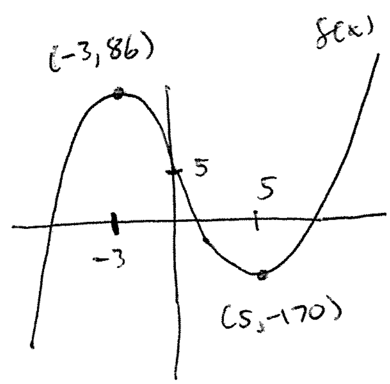


$x = 5$ is a local min

$x = -3$ " " " max

f increasing on $(-\infty, -3) \cup (5, \infty)$

f decreasing " $(-3, 5)$



sketch not drawn to scale

The second derivative can also be used to determine the type of critical point. If $x = c$ is a critical point and $f''(c) > 0$, then $x = c$ is a local minimum because the function is concave up at $x = c$. (The sign of the derivative goes from $-$, to 0 at $x = c$, then becomes $+$ for $x > c$, so by the first derivative test, c is a local min.)

Second Derivative Test: Suppose that $x = c$ is a critical point of f.

$f''(c) > 0$	$\Longrightarrow$	c is a local min
$f''(c) < 0$	$\Longrightarrow$	c is a local max
$f''(c) = 0$	$\Longrightarrow$	test is inconclusive

Note: If c is a critical point and $f''(c) = 0$, then c may either be a local max, a local min, or neither. For example, $x = 0$ is a critical point for all three of the following functions: $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$. We also have $f''(0) = g''(0) = h''(0) = 0$, so the second derivative test is inconclusive in all three cases. However, by graphing each function, it is straight-forward to see that $x = 0$ is a local min for f, a local max for g, and neither for h (it's actually an inflection point for h).

Example 3: For the function $f(x) = \dfrac{x^2 - 8x}{x + 1}$, find and simplify $f'(x)$ and $f''(x)$. Then find the critical points of f and use the second derivative test to determine whether each critical point is a local max, local min, or neither.

Quotient Rule: $f'(x) = \dfrac{(x+1)(2x-8) - (x^2-8x)(1)}{(x+1)^2} = \dfrac{2x^2-6x-8-x^2+8x}{(x+1)^2}$

$$= \frac{x^2+2x-8}{(x+1)^2} = \frac{(x+4)(x-2)}{(x+1)^2}$$

$f'(x) = 0$ when numerator $= 0$ so $x = -4, 2$ are the critical pts

$$f''(x) = \frac{(x+1)^{\cancel{2}\,1}(2x+2) - (x^2+2x-8)\cdot 2\cancel{(x+1)}\cdot 1}{(x+1)^{\cancel{4}\,3}}$$

Cancel $x+1$ from each term

$$= \frac{(x+1)(2x+2) - 2(x^2+2x-8)}{(x+1)^3} = \frac{\cancel{2x^2} + \cancel{4x} + 2 - \cancel{2x^2} - \cancel{4x} + 16}{(x+1)^3}$$

$$= \frac{18}{(x+1)^3}.$$

So $f''(-4) = \dfrac{18}{-27} < 0 \Rightarrow$ $x = -4$ local max

$f''(2) = \dfrac{18}{27} > 0 \Rightarrow$ $x = 2$ local min

Example 4: Consider the function $g(x) = x^4 - 4x^3$. Find and classify all critical points. Find any inflection points. Use the first and second derivatives to sketch the graph of g.

$g'(x) = 4x^3 - 12x^2 = 4x^2(x-3) \Rightarrow x = 0, 3$ are critical points

$g''(x) = 12x^2 - 24x = 12x(x-2)$

$x = 3$ is a min

but $x = 0$ is an inflection pt.

decr. decr. incr.

$-$ $-$ $+$ f'

0 3

$\Downarrow$ $\Downarrow$

nothing min

$g(0) = 0$, $g(2) = -16$, $g(3) = -27$

$g(4) = 0$.

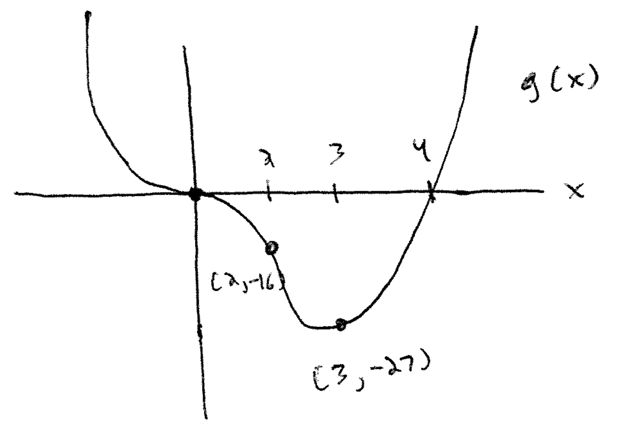
up down up

$+$ $-$ $+$ f''

0 2

$x = 0$ and $x = 2$

are each inflection points.



Start

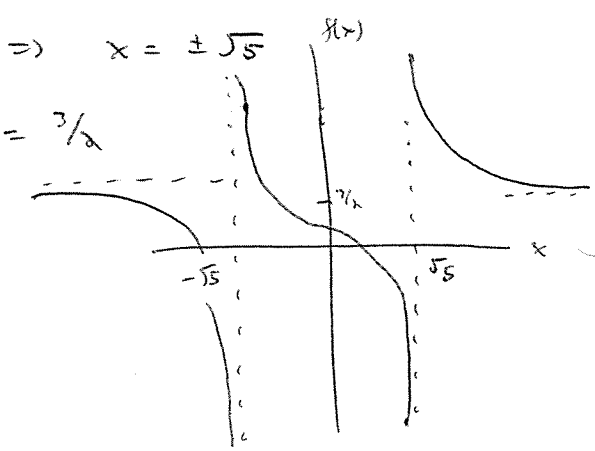
Functions with Asymptotes:

Recall that $f(x)$ has a **vertical asymptote** at $x = b$ if either $\lim_{x\to b+} f(x) = \pm\infty$ or $\lim_{x\to b-} f(x) = \pm\infty$. These limits need to be in agreement with the information obtained from the first and second derivatives. A function $f(x)$ has a **horizontal asymptote** at $y = k$ if $\lim_{x\to\infty} f(x) = k$ or $\lim_{x\to-\infty} f(x) = k$. In the first case, the asymptote occurs to the far right of the graph, while in the second case, the asymptote appears to the far left.

Example 5: Identify the vertical and horizontal asymptotes of the function $f(x) = \dfrac{3x^2 + 5x - 7}{2x^2 - 10}$.

vertical: $2x^2 - 10 = 0 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

Horizontal: $\lim_{x\to\infty} f(x) = 3/2 \Rightarrow y = 3/2$



Example 6: Identify the vertical and horizontal asymptotes of $g(x) = \dfrac{5x-3}{2x+1}$ and then use the first and second derivatives of g to sketch its graph.

vert: $x = -1/2$

horiz: $y = 5/2$

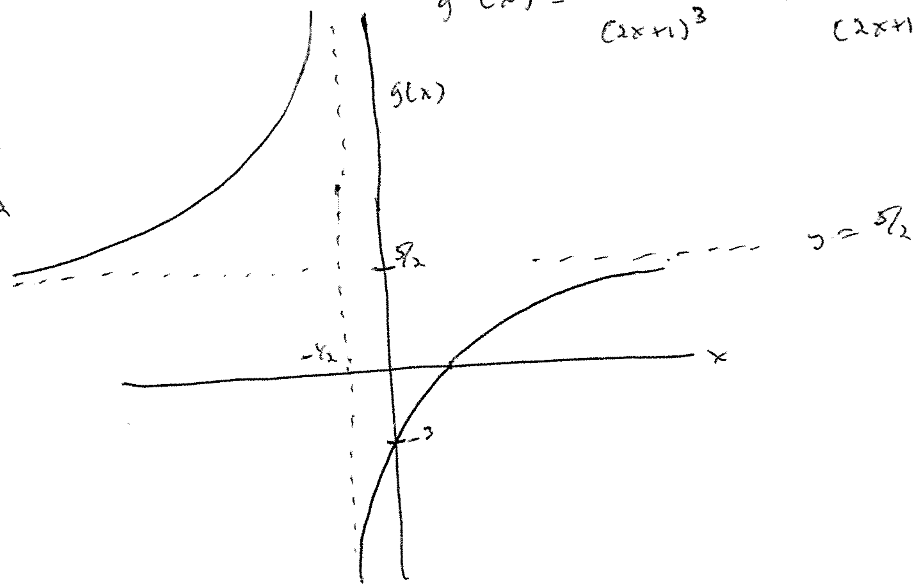
$$g'(x) = \frac{(2x+1)5 - (5x-3)\cdot 2}{(2x+1)^2} = \frac{11}{(2x+1)^2} > 0$$

$$g''(x) = \frac{-22 \cdot 2}{(2x+1)^3} = \frac{-44}{(2x+1)^3}$$

g always increasing (except for $x = -1/2$)

g concave up : $x < -1/2$

g " down: $x > -1/2$



Example 7: Carefully find and simplify the first and second derivatives of $f(x) = xe^{-x}$. Use this information to find the critical and inflection points of f. Sketch the graph of f. Are there any asymptotes? Yes

$f'(x) = e^{-x} + x \cdot (-e^{-x}) = e^{-x}(1-x)$

$f''(x) = -e^{-x}(1-x) + e^{-x}(-1) = e^{-x}(x-2)$

$x = 1$ is a crit. pt. (max)

f': $+$ 0 $-$ (at 1)

f'': $-$ 0 $+$ (at 2); down, up

$x = 2$ is an infl. pt.

$e^{-x} > 0 \quad \forall x$

$f(x) = \dfrac{x}{e^x}$

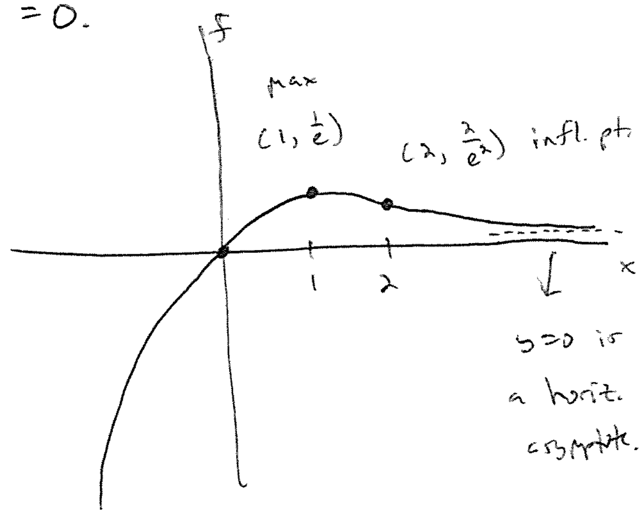
$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

L'Hopital's Rule

No vert. asymptotes.

Horiz. $\displaystyle\lim_{x\to\infty} f(x) = 0$

$\Rightarrow$ $y = 0$ is a horiz. asymp.



Example 8: Use the first and second derivative to sketch the graph of $y = \sin x + \frac{1}{2}x$ over the interval $[0, 2\pi]$. Identify all critical points and inflection points on your graph.

$y' = \cos x + \frac{1}{2}$

$y'' = -\sin x$

$\cos x + \frac{1}{2} = 0 \Rightarrow \cos x = -\frac{1}{2}$

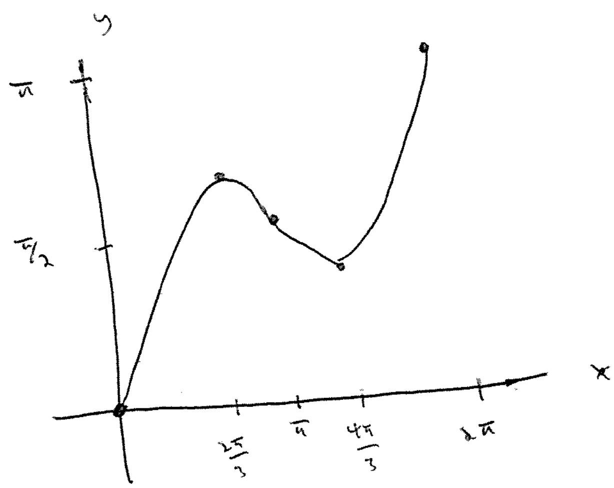
$\Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$ are crit. pts

y': $+$ (incr.) on $(0, \frac{2\pi}{3})$, 0 at $\frac{2\pi}{3}$, $-$ (decr.) on $(\frac{2\pi}{3}, \frac{4\pi}{3})$, 0 at $\frac{4\pi}{3}$, $+$ (incr.) on $(\frac{4\pi}{3}, 2\pi)$

max at $\frac{2\pi}{3}$, min at $\frac{4\pi}{3}$

$-\sin x = 0 \Rightarrow x = \pi$ is an inflection pt

y'': $-$ (down) on $(0, \pi)$, 0 at π, $+$ (up) on $(\pi, 2\pi)$



$f(0) = 0$

$f(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{\pi}{3} \approx 1.91$

$f(\pi) = \frac{\pi}{2} \approx 1.57$

$f(\frac{4\pi}{3}) = -\frac{\sqrt{3}}{2} + \frac{2\pi}{3} \approx 1.23$

$f(2\pi) = \pi \approx 3.14$