MATH 135 Calculus 1, Spring 2016

Derivatives of Trig Functions and the Chain Rule

3.6 Trig Functions

The following key derivatives can be proven using the definition of the derivative and the trig identities $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$ and $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$.

Theorem 0.1 *Assume that x is measured in radians.*

$$\boxed{\frac{d}{dx}(\sin x) \;=\; \cos x \qquad \textit{and} \qquad \frac{d}{dx}(\cos x) \;=\; -\sin x\,.} \tag{1}$$

To prove the first formula, let $f(x) = \sin x$. Then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \cdot \sin h}{h} \\
&= \lim_{h\to 0} \sin x \cdot \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \;=\; \cos x,
\end{aligned}$$

where we have made use of the two important trig limits $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\displaystyle\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$. A similar proof can be used to show that the derivative of $\cos x$ is $-\sin x$.

The derivatives of the other four trig functions can now be found using the quotient rule. For instance, we have that $\displaystyle\frac{d}{dx}(\tan x) \;=\; \sec^2 x$ because

$$\frac{d}{dx}(\tan x) \;=\; \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \;=\; \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \;=\; \frac{1}{\cos^2 x} \;=\; \left(\frac{1}{\cos x}\right)^2 \;=\; \sec^2 x.$$

Exercise 0.2 *Use the quotient rule to show that* $\displaystyle\frac{d}{dx}(\cot x) \;=\; -\csc^2 x$.

Exercise 0.3 *Show that* $\dfrac{d}{dx}(\sec x) = \sec x \cdot \tan x.$

Exercise 0.4 *If* $g(x) = x^3 \sin x$*, find and simplify* $g'(x)$ *and* $g''(x)$*.*

3.7 The Chain Rule

One of the most important differentiation rules is the **Chain Rule**, used to take the derivative of the composition of two functions. The chain rule allows us to differentiate complicated functions that can be broken down into the composition of simpler functions.

Theorem 0.5 (Chain Rule) *If* $f(x)$ *and* $g(x)$ *are differentiable functions, then so is the composition* $f(g(x))$*. We have*

$$\boxed{\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)\,.} \tag{2}$$

In words, Formula (2) states that the derivative of the composition of two functions is given by the derivative of the outside function **evaluated at the inside function** times the derivative of the inside function.

Let us check the Chain Rule on a simple example. Suppose that $h(x) = (x^4+1)^2$. The outside function is $f(x) = x^2$, since the outer operation is squaring, and the inside function is $g(x) = x^4+1$. Then we have $h(x) = f(g(x))$. Since $f'(x) = 2x$ and $g'(x) = 4x^3$, the Chain Rule gives

$$h'(x) = 2(x^4+1) \cdot 4x^3 = 8x^3(x^4+1) = 8x^7 + 8x^3.$$

To check this, we can first expand $h(x)$ by writing

$$h(x) = (x^4+1)^2 = (x^4+1)(x^4+1) = x^8 + 2x^4 + 1.$$

By the Power Rule, we have $h'(x) = 8x^7 + 8x^3$, which confirms the calculation above via the Chain Rule.

Exercise 0.6 *Use the Chain Rule to find $h'(x)$ if $h(x) = \sin(x^5 - 6x)$. What is the outside function $f(x)$ and the inside function $g(x)$?*

Exercise 0.7 *If $F(x) = \sqrt{x^4 + 3}$, use the Chain Rule to find and simplify $F'(x)$.*

Leibniz Notation

It is instructive to write the Chain Rule using Leibniz notation. Suppose that $y = f(x)$ and $x = g(t)$ (y is a function of x and x, in turn, is a function of t). Then $y = f(g(t))$ and by the Chain Rule, $dy/dt = f'(g(t)) \cdot g'(t)$ or more simply

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}\ .} \tag{3}$$

The nice aspect of Formula (3) is that we can visualize the dx's canceling out to help remember the Chain Rule. In this form of the Chain Rule, the derivative of the outside is given by dy/dx and the derivative of the inside is given by dx/dt.

Exercise 0.8 *If $y = e^{x^2}$, calculate $\dfrac{dy}{dx}$. If $z = e^{\tan t}$, calculate $\dfrac{dz}{dt}$.*

Hint: *In both cases, the outside function is the same.*

Exercise 0.9 *Find a general formula for $\dfrac{dy}{dx}$ when $y = e^{u(x)}$.*

Sometimes we need to apply the Chain Rule multiple times.

Exercise 0.10 *Find and simplify* $G'(x)$ *if* $G(x) = \sin(\sqrt{x^2+2}) + e^{\cos(4x)}$.

Exercise 0.11 *Calculate* $\frac{d}{dt}\left(\sin(\cos(e^{6t}))\right)$.

Exercise 0.12 *Find and simplify* $F'(x)$ *if* $F(x) = \sin(x^2)\cos(x^2)$. *Then find the equation of the tangent line at the point* $a = \sqrt{\pi}$.

Exercise 0.13 *Suppose that* $h(x) = f(g(x))$ *and that* $f'(3) = 4, f(3) = 2, f'(6) = -1, g(3) = 6$ *and* $g'(3) = 7$. *What is* $h'(3)$*?*