MATH 135 Calculus 1, Spring 2016

Worksheet for Section 3.1

3.1 Definition of the Derivative

This section introduces the **derivative**, one of the most important concepts in all of mathematics and science. It is the foundation of calculus.

What is the derivative?

The derivative of a function is the **slope** of the tangent line to the graph of the function. For "nice" functions, the derivative exists, but it is important to realize that sometimes the derivative may not exist. Another important interpretation of the derivative is that it represents a **rate of change**. For example, if $s(t)$ represents the position of an object at time t, then the derivative, denoted $s'(t)$, is the instantaneous velocity at time t. We can think of velocity as measuring the rate of change of distance with respect to time. (The units are revealing here: miles per hour or meters per second.)

Recall that the instantaneous velocity was found by taking the limit of average velocities over smaller and smaller time periods. Geometrically, this was visualized as taking the limit of the slopes of the secant lines to produce the slope of the tangent line. Generalizing this idea, we have the following critical definition:

Definition 0.1 *The* **derivative** *of $f(x)$ at the point $x = a$, denoted by $f'(a)$ (read as "f prime of a"), is given by*

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.} \tag{1}$$

This limit may or may not exist. If the limit exists, we say that the function is **differentiable** *at $x = a$.*

The fraction in Equation (1) is called the **difference quotient**. Note that if we evaluate the difference quotient at $x = a$ we get $\frac{0}{0}$, a familiar indeterminate form that can be ANYTHING. When computing the derivative using Equation (1) we typically must simplify the difference quotient or use a calculator to evaluate the limit.

Exercise 0.2 *Use Equation (1) to compute the derivative of $f(x) = x^2 - 3x$ at the point $x = 1$. In other words, compute $f'(1)$. What is the equation of the tangent line to f at $x = 1$? (see Figure 1)*

Set $a=1$ in formula (1).

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \frac{x^2 - 3x - (-2)}{x-1}$$

$$= \lim_{x \to 1} \frac{x^2 - 3x + 2}{x-1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x-2)}{\cancel{x-1}}$$

$$= \lim_{x \to 1} x - 2 = 1 - 2 = -1$$

slope of tangent line to graph at $x=1$, $y=-2$

slope is -1

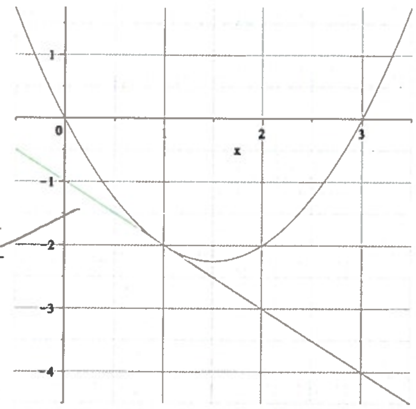


Figure 1: The graph of the function $f(x) = x^2 - 3x$ along with the tangent line at the point $x = 1$.

Another equivalent definition of the derivative comes from making the substitution $h = x - a$. The difference quotient in Equation (1) then transforms to

$$\frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h},$$

which has the advantage of a simpler denominator. Since $h \to 0$ as $x \to a$, we have the following equivalent definition of the derivative.

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.} \qquad (2)$$

You should MEMORIZE both Equations (1) and (2). Equation (2) is typically easier to apply because the denominator is just an h, but this is not always the case. Remember that a is just a constant, the point at which you are computing the slope of the tangent line. The general steps for computing the derivative using a limit definition are:

1. Find the difference quotient $\dfrac{f(a+h) - f(a)}{h}$ or $\dfrac{f(x) - f(a)}{x - a}$.
2. Simplify the difference quotient. This involves algebra such as factoring, adding fractions, multiplying by the conjugate, etc.
3. Take the limit as $h \to 0$ or as $x \to a$.

$a = 1$

Exercise 0.3 *Using Equation (2), compute $f'(1)$ for $f(x) = x^2 - 3x$ and check that you obtain the same answer as in Exercise 0.2.*

Plug in $a = 1$ into equation (2).

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 3(1+h) - (-2)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 - 3 - 3h + 2}{h} = \lim_{h \to 0} \frac{h^2 - h}{h}$$

h's cancel

$$= \lim_{h \to 0} \frac{\cancel{h}(h-1)}{\cancel{h}} = 0 - 1 = -1 \checkmark$$

$m = -4 = \text{slope}$

Exercise 0.4 *Find $f'(3)$ if $f(x) = -4x+7$. Why do you expect to get this answer? What is $f'(-419)$?*

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{-4(3+h) + 7 - (-5)}{h}$$

$$= \lim_{h \to 0} \frac{-12 - 4h + 12}{h} = \lim_{h \to 0} \frac{-4\cancel{h}}{\cancel{h}} = \lim_{h \to 0} -4 = -4$$

This makes sense b/c $f(x)$ is a linear function so its graph is a line. The tangent line to a line is the line itself. So $f'(a) = -4$ for any a since $m = -4$.

Exercise 0.5 *Using a limit definition, find the slope of the tangent line to the function $f(x) = \frac{2}{x} + 1$ at the point $x = 2$.*

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{2}{x} + 1 - \left(\frac{2}{2} + 1\right)}{x - 2}$$

LCD $= x$

$$= \lim_{x \to 2} \frac{\frac{2}{x} + 1 - 1 - 1}{x - 2} = \lim_{x \to 2} \frac{\frac{2}{x} - 1}{x - 2} = \lim_{x \to 2} \frac{\frac{2}{x} - \frac{x}{x}}{x - 2}$$

$$= \lim_{x \to 2} \frac{\frac{2-x}{x}}{\frac{x-2}{1}} = \lim_{x \to 2} \frac{\overset{(-1)}{\cancel{2-x}}}{x} \cdot \frac{1}{\cancel{x-2}} = \lim_{x \to 2} \frac{-1}{x} = -\frac{1}{2}$$

Exercise 0.6 *Find $f'(4)$ if $f(x) = \sqrt{x}$. What is the equation of the tangent line to f at the point $x = 4$?* **Hint:** *Multiply the top and bottom of the difference quotient by the conjugate.*

Conjugate

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{(\sqrt{4+h} + 2)}{(\sqrt{4+h} + 2)}$$

$$= \lim_{h \to 0} \frac{4 + h - 4}{h(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{\cancel{h}^{\,1}}{\cancel{h}(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2}$$

don't distribute

$$= \frac{1}{\sqrt{4} + 2} = \frac{1}{2+2} = \frac{1}{4}$$

Tangent line: use $y = mx + b$ with $m = 1/4$

$\Rightarrow$ $y = 1/4\, x + b$. Since $f(4) = 2$, $(4,2)$ is on the line.

$\therefore$ $2 = 1/4(4) + b$ or $2 = 1 + b \Rightarrow b = 1$. $y = \frac{1}{4}x + 1$

Exercise 0.7 *Using a limit definition, find $g'(-2)$ if $g(x) = x^3 - 4x$.*

$$g'(-2) = \lim_{x \to -2} \frac{g(x) - g(-2)}{x - (-2)} = \lim_{x \to -2} \frac{x^3 - 4x - (-8+8)}{x+2}$$

$$= \lim_{x \to -2} \frac{x^3 - 4x}{x+2} = \lim_{x \to -2} \frac{x(x^2-4)}{x+2}$$

$$= \lim_{x \to -2} \frac{x\cancel{(x+2)}(x-2)}{\cancel{x+2}} = \lim_{x \to -2} x(x-2)$$

$$= -2(-4) = 8$$

$$g'(-2) = 8$$

Exercise 0.8 *Find $f'(0)$ if $f(x) = e^x$. Why does your answer make sense given the definition of e?*
Hint: *You will need to use a calculator to do the limit.*

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 1}{h}$$

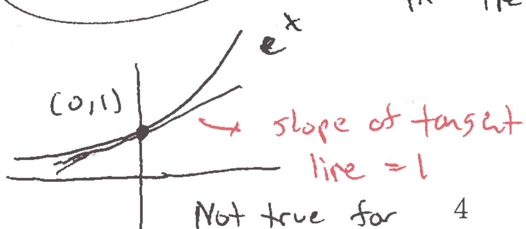
$$= \lim_{h \to 0} \frac{e^h - 1}{h} \longrightarrow$$ no algebra can be done to simplify this limit

Plug in h-values very close to $h=0$.

$h = 0.001$ gives 1.0005 $\Rightarrow$ limit is 1.
$h = -0.001$ " 0.9995

$\therefore$ $f'(0) = 1$

This is actually how e was defined in the first place.



Not true for b^x if $b \neq e$.

e^x
$\swarrow$
e is the unique base that has a slope of 1 at $x=0$.