MATH 135-04 Calculus 1

Exam #3 SOLUTIONS **December 3, 2015** **Prof. G. Roberts**

1. Calculate the derivative of each function. **Simplify** your answer as best as possible. (30 pts.)

(a) $f(x) = \dfrac{4}{x^4} + 4^x + e^4$

Answer: First, to avoid using the quotient rule, we write $f(x) = 4x^{-4} + 4^x + e^4$. Then, using the power rule and the rule for differentiating an exponential function, we have

$$f'(x) \;=\; -16x^{-5} + 4^x \cdot \ln 4 \;=\; \frac{-16}{x^5} + (\ln 4) \cdot 4^x.$$

Note that e^4 is just a constant, so its derivative is 0.

(b) $g(x) = (\sec(3x) + e^{3x})^{5/3}$

Answer: Using the chain rule twice, we have

$$\begin{aligned} g'(x) \;&=\; \frac{5}{3}\left(\sec(3x) + e^{3x}\right)^{2/3} \cdot \left(\sec(3x)\tan(3x) \cdot 3 + 3e^{3x}\right) \\ &=\; 5\left(\sec(3x) + e^{3x}\right)^{2/3} \cdot \left(\sec(3x)\tan(3x) + e^{3x}\right). \end{aligned}$$

(c) $F(t) = e^{\tan^{-1}(\sqrt{t})}$

Answer: Using the chain rule twice, we have

$$F'(t) \;=\; e^{\tan^{-1}(\sqrt{t})} \cdot \frac{1}{1 + (\sqrt{t})^2} \cdot \frac{1}{2}t^{-1/2} \;=\; \frac{e^{\tan^{-1}(\sqrt{t})}}{2\sqrt{t}\,(1+t)}.$$

(d) $G(\theta) = \ln(\,\ln(\cos\theta)\,)$

Answer: Using the chain rule twice, we have

$$G'(\theta) \;=\; \frac{1}{\ln(\cos\theta)} \cdot \frac{1}{\cos(\theta)} \cdot -\sin(\theta) \;=\; \frac{-\tan\theta}{\ln(\cos\theta)}.$$

(e) $y = x^{\tan x}$ *Hint:* Use logarithmic differentiation.

Answer: Take the natural log of both sides to obtain

$$\ln y \;=\; \ln x^{\tan x} \;=\; \tan x \cdot \ln x.$$

Next we differentiate both sides with respect to x, using implicit differentiation on the left-hand side and the product rule on the right-hand side. This gives

$$\frac{1}{y} \cdot \frac{dy}{dx} \;=\; \sec^2 x \cdot \ln x \;+\; \tan x \cdot \frac{1}{x}.$$

Multiplying both sides by y and substituting in for y gives the answer

$$\frac{dy}{dx} \;=\; y\left(\sec^2 x \cdot \ln x \;+\; \frac{\tan x}{x}\right) \;=\; x^{\tan x}\left(\sec^2 x \cdot \ln x \;+\; \frac{\tan x}{x}\right).$$

2. For the equation below, use implicit differentiation to calculate dy/dx. (12 pts.)

$$\sin(xy) + e^{y^3} = \sin^{-1} x - 4\cos y$$

Answer: Differentiating each side with respect to x and treating $y = y(x)$ as a function of x, we have, by the chain rule,

$$\cos(xy) \cdot \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + e^{y^3} \cdot 3y^2 \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + 4\sin y \cdot \frac{dy}{dx}.$$

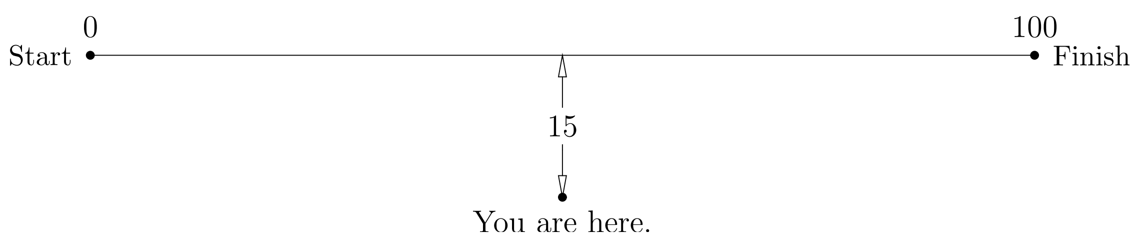
Grouping all terms with dy/dx together on one side of the equation yields

$$x\cos(xy) \cdot \frac{dy}{dx} + 3y^2 e^{y^3} \cdot \frac{dy}{dx} - 4\sin y \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - y\cos(xy)$$

which gives, after factoring out the dy/dx on the left-hand side,

$$\frac{dy}{dx} = \frac{\frac{1}{\sqrt{1-x^2}} - y\cos(xy)}{x\cos(xy) + 3y^2 e^{y^3} - 4\sin y}.$$

3. You are at a Holy Cross track meet watching the 100 meter dash, sitting in the bleachers 15 meters back from the halfway point of the race. Your friend Alli the speedster is running down the track at a constant speed of 8.5 meters per second. How fast is the distance between you and Alli changing when she is 80 meters into the race? Round your answer to one decimal place. (13 pts.)



Answer: 7.6 meters/sec

Draw a right triangle with the height equal to 15, the base equal to x, and the hypotenuse equal to z. The right angle of this triangle occurs at the midway point of the race, 50 meters along the track. The hypotenuse z is the distance between you and Alli. Both the quantities x and z are changing over time, but the distance 15 is fixed because you are not moving. When Alli is 80 meters into the race, we have $x = 30$. We want to find dz/dt when $x = 30$.

By the Pythagorean Theorem, we have $x^2 + 15^2 = z^2$. Differentiating this equation with respect to t yields $2x \cdot \frac{dx}{dt} = 2z \cdot \frac{dz}{dt}$ which simplifies to

$$\frac{dz}{dt} = \frac{x}{z} \cdot \frac{dx}{dt}.$$

We are given that $dx/dt = 8.5$. To find z when $x = 30$, use the Pythagorean Theorem to obtain $z = \sqrt{30^2 + 15^2} = \sqrt{1125} = 15\sqrt{5} \approx 33.541$. Therefore, we find that

$$\frac{dz}{dt} = \frac{30}{15\sqrt{5}} \cdot 8.5 = \frac{17}{\sqrt{5}} \approx 7.6 \text{ meters/sec.}$$

4. Find the linearization $L(x)$ of the function $f(x) = \sqrt[3]{x}$ at the point $a = 27$. Use the linearization to approximate the value of $\sqrt[3]{24}$ (to four decimal places) and then use a calculator to compute the percentage error in your approximation (to three decimal places). (12 pts.)

Answer: We use the formula $L(x) = f(a) + f'(a)(x-a)$ with $a = 27$. Since $f(x) = x^{1/3}$, we have that $f'(x) = \frac{1}{3}x^{-2/3}$. Thus, $f(27) = \sqrt[3]{27} = 3$ and

$$f'(27) = \frac{1}{3} \cdot 27^{-2/3} = \frac{1}{3} \cdot \frac{1}{27^{2/3}} = \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}.$$

It follows that

$$L(x) = 3 + \frac{1}{27}(x - 27) = 2 + \frac{1}{27}x.$$

To approximate $\sqrt[3]{24}$, we plug in $x = 24$ into our linearization:

$$L(24) = 2 + \frac{24}{27} = \frac{26}{9} \approx 2.8889.$$

The percentage error is then

$$100 \cdot \left| \frac{2.8889 - \sqrt[3]{24}}{\sqrt[3]{24}} \right| \approx 0.153\%.$$

5. Find the absolute maximum and absolute minimum of the function

$$f(x) = 5\sqrt{1 + x^2} - 3x$$

over the interval $[0, \sqrt{3}]$. Give the maximum and minimum function values as well as the x-values where they occur. (13 pts.)

Answer: The absolute maximum is 5 at $x = 0$ and the absolute minimum is 4 at $x = 3/4$. First, we find the critical points of f on the interval $[0, \sqrt{3}]$. Write $f(x) = 5(1 + x^2)^{1/2} - 3x$ and use the chain rule. This gives

$$f'(x) = \frac{5}{2}\left(1 + x^2\right)^{-1/2} \cdot 2x - 3 = 5x(1 + x^2)^{-1/2} - 3 = \frac{5x}{\sqrt{1 + x^2}} - 3.$$

To solve $f'(x) = 0$, we write

$$\frac{5x}{\sqrt{1 + x^2}} = \frac{3}{1}$$

and cross multiply. This gives $5x = 3\sqrt{1 + x^2}$. Squaring both sides yields $25x^2 = 9(1 + x^2)$ or $25x^2 = 9 + 9x^2$. Solving for x^2 gives $x^2 = 9/16$ so that $x = \pm 3/4$. However, since $x = -3/4$ is not in the domain $[0, \sqrt{3}]$, we ignore it.

Plugging the endpoints and the critical point $x = 3/4$ into the function f gives

$$f(0) = 5, f(3/4) = 4, \text{ and } f(\sqrt{3}) = 10 - 3\sqrt{3} \approx 4.8.$$

Thus, 5 is the absolute max and 4 is the absolute min.

6. Some final conceptual questions. (20 pts.)

(a) When describing the recent price of VK stock, a market analyst states, "Although the price of VK stock continues to decline, it is declining at a slower rate. It might **not** be wise to completely sell off the stock." If $p(t)$ is the price of VK stock at time t, what are the signs (positive, negative, or zero) of $p'(t)$ and $p''(t)$?

Answer: $p'(t) < 0$ and $p''(t) > 0$.

Since the price is decreasing, we have $p'(t) < 0$. But since the decline is slowing down, the slopes are getting less negative, that is, the derivative is increasing and the curve is concave up. Thus, $p''(t) > 0$.

(b) The total dollar cost of producing x plasma television sets is given by the function

$$C(x) = 150 - 90x - 0.07x^2 + 0.001x^3.$$

Give the marginal cost function and use it to estimate the cost of producing the 401st television set.

Answer: The marginal cost function is $C'(x) = -90 - 0.14x + 0.003x^2$. Recall that the marginal cost function estimates the cost of producing the next item (the $(x+1)$th item). Thus, to estimate the cost of producing the 401st television set, we compute $C'(400) = \$334$.

(c) Suppose that $G(x) = f(x^2)$ and that $f'(4) = 5$, $f''(4) = -1$. Find $G''(2)$.

Answer: -6.

By the chain rule, we have $G'(x) = f'(x^2) \cdot 2x$. To find $G''(x)$ we use the product rule and the chain rule. We have

$$G''(x) = f''(x^2) \cdot 2x \cdot 2x + f'(x^2) \cdot 2 = 4x^2 \cdot f''(x^2) + 2f'(x^2).$$

Plugging $x = 2$ into the last equation gives

$$G''(2) = 16 \cdot f''(4) + 2 \cdot f'(4) = 16 \cdot (-1) + 2 \cdot 5 = -6.$$