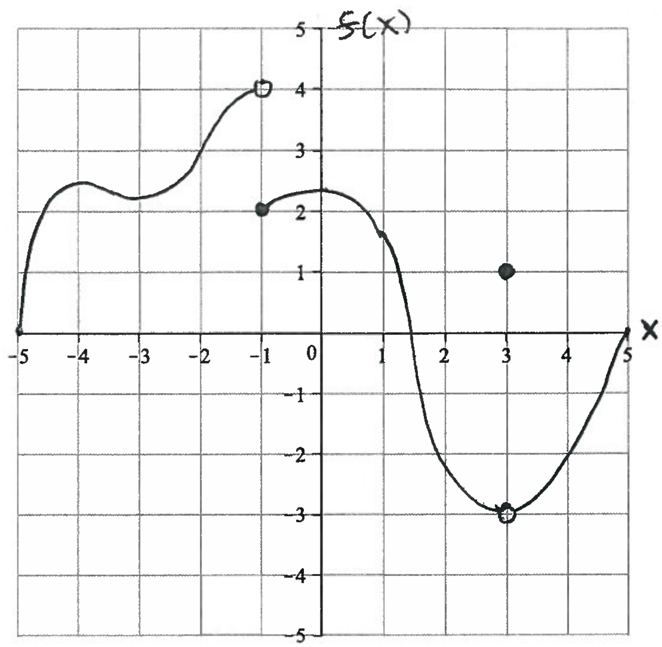
# MATH 135-04 Calculus I

**Exam #2 SOLUTIONS** **November 5, 2015** **Prof. G. Roberts**

1. The graph of $f(x)$ is shown below. Use it to answer each of the following questions. (15 pts.)




**(a)** Evaluate $\lim_{x\to -1^-} f(x)$. **Answer:** 4

**(b)** Evaluate $\lim_{x\to -1^+} f(x)$ **Answer:** 2

**(c)** Is $f$ left-continuous, right-continuous, or continuous at $x = -1$? Explain.

**Answer:** $f$ is right-continuous at $x = -1$ because the function value equals the right-hand limit, that is, $\lim_{x\to -1^+} f(x) = f(-1) = 2$.

**(d)** Evaluate $\lim_{x\to 3} f(x)$ **Answer:** $-3$

**(e)** How should $f(3)$ be redefined to remove the discontinuity at $x = 3$?

**Answer:** Set $f(3) = -3$. Since the limit at $x = 3$ is $-3$, to be continuous, we should redefine $f(3)$ to be $-3$.

2. Evaluate each of the following limits, if they exist. Note that $\infty$ or $-\infty$ are acceptable answers. You must show work (e.g., algebra) to receive full credit. (5 pts. each)

**(a)** $\lim_{x\to \pi} \sqrt{2\cos^2 x + 3}$

**Answer:** $\sqrt{5}$. Using the fact that the function is continuous at $x = \pi$, we simply plug in $x = \pi$ to obtain $\sqrt{2\cos^2(\pi) + 3} = \sqrt{2(-1)^2 + 3} = \sqrt{5}$.

**(b)** $\lim_{x\to 5} \dfrac{2x^2 - 7x - 15}{x^2 - 25}$

**Answer:** 13/10. Factor and cancel. We have

$$\lim_{x\to 5} \frac{2x^2 - 7x - 15}{x^2 - 25} = \lim_{x\to 5} \frac{(2x+3)(x-5)}{(x+5)(x-5)} = \lim_{x\to 5} \frac{2x+3}{x+5} = \frac{13}{10}.$$

**(c)** $\lim_{\theta\to 0} \dfrac{\tan\theta}{3\theta}$

**Answer:** 1/3. Rewrite the limit as a product and use the fact that the limit of a product equals the product of the limits. We have

$$\lim_{\theta\to 0} \frac{\tan\theta}{3\theta} = \lim_{\theta\to 0} \frac{\frac{\sin\theta}{\cos\theta}}{\frac{3\theta}{1}} = \lim_{\theta\to 0} \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{3\theta} = \frac{1}{3}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta} = \frac{1}{3}\cdot 1\cdot\frac{1}{1} = \frac{1}{3}.$$

**(d)** $\lim_{x\to -3}\left(\dfrac{1}{x+3}+\dfrac{6}{x^2-9}\right)$

**Answer:** $-1/6$. Add the fractions and then simplify. We have

$$\begin{aligned}
\lim_{x\to -3}\left(\frac{1}{x+3}+\frac{6}{x^2-9}\right) &= \lim_{x\to -3}\left(\frac{x-3}{(x+3)(x-3)}+\frac{6}{(x+3)(x-3)}\right)\\
&= \lim_{x\to -3}\frac{x+3}{(x+3)(x-3)}\\
&= \lim_{x\to -3}\frac{1}{x-3} = -\frac{1}{6}\,.
\end{aligned}$$

**(e)** $\lim_{x\to\infty} \tan^{-1}\left(\dfrac{x^4-1}{5x^3+x^2+2}\right)$

**Answer:** $\pi/2$. First take the limit of the fraction inside the parentheses. Since the numerator has the higher power, and since the leading coefficients of the numerator and denominator are both positive, the fraction is heading toward $+\infty$. Now we need to compute $\tan^{-1}(\infty)$, or more specifically, $\lim_{x\to\infty}\tan^{-1}x$. Recall from its graph that $\tan^{-1}x$ has a horizontal asymptote to the right at $y=\pi/2$. Thus, the limit we seek is equal to $\pi/2$.

3. Use the Intermediate Value Theorem to prove that the equation $e^{x^2}=x+2$ has a solution in the interval $[1,2]$. (10 pts.)

**Answer:** First, we define the function $f(x)=e^{x^2}-x-2$. We would like to find a number $c$ between 1 and 2 such that $f(c)=0$, as then we would have $e^{c^2}-c-2=0$, which is equivalent to $e^{c^2}=c+2$ (the equation we would like to solve). Note that $f$ is a continuous function since it is the difference of two continuous functions, an exponential function and a linear function.

We compute that $f(1)=e-3\approx -0.28<0$ while $f(2)=e^4-4\approx 50.6>0$. By the Intermediate Value Theorem, since $f(1)<0$ and $f(2)>0$, there exists a number $c$ between 1 and 2 such that $f(c)=0$, as desired.

4. **(a)** State one of the two limit definitions for the derivative of a function $f(x)$ at the point $x=a$. (3 pts.)

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

**(b)** Use your limit definition from part **(a)** to find $f'(1)$ where $f(x) = \sqrt{3x+1}$. (9 pts.)

**Answer:** $f'(1) = 3/4$.

*Method 1*: Using the $\lim\limits_{h\to 0}$ definition, we have

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(1+h)+1}-2}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3+3h+1}-2}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3h+4}-2}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{3h+4}-2)}{h} \cdot \frac{(\sqrt{3h+4}+2)}{(\sqrt{3h+4}+2)} \\
&= \lim_{h\to 0} \frac{3h+4-4}{h(\sqrt{3h+4}+2)} \\
&= \lim_{h\to 0} \frac{3}{(\sqrt{3h+4}+2)} \\
&= \frac{3}{\sqrt{4}+2} = \frac{3}{4}.
\end{aligned}$$

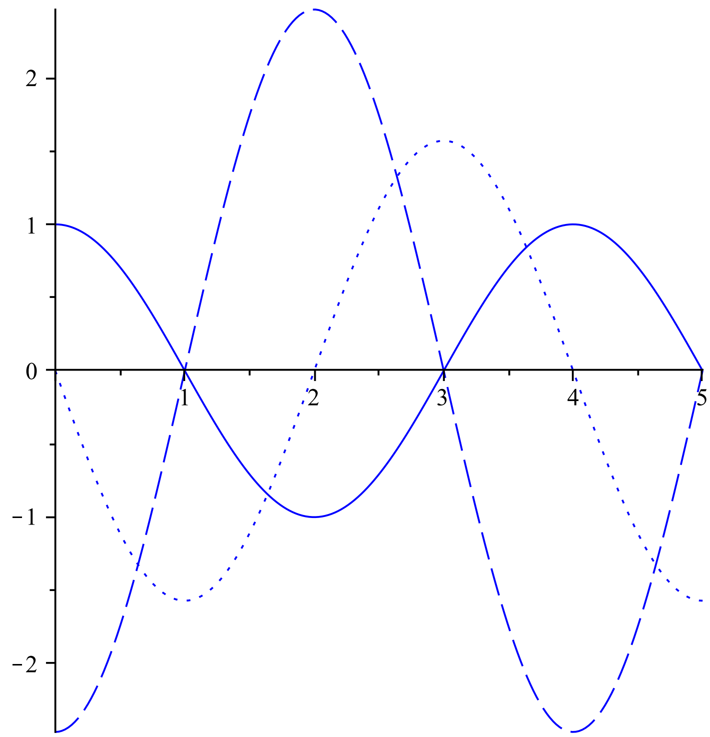
*Method 2*: Using the $\lim\limits_{x\to a}$ definition, we have

$$\begin{aligned}
f'(1) &= \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} \\
&= \lim_{x\to 1} \frac{\sqrt{3x+1}-2}{x-1} \\
&= \lim_{x\to 1} \frac{(\sqrt{3x+1}-2)}{x-1} \cdot \frac{(\sqrt{3x+1}+2)}{(\sqrt{3x+1}+2)} \\
&= \lim_{x\to 1} \frac{3x+1-4}{(x-1)(\sqrt{3x+1}+2)} \\
&= \lim_{x\to 1} \frac{3x-3}{(x-1)(\sqrt{3x+1}+2)} \\
&= \lim_{x\to 1} \frac{3(x-1)}{(x-1)(\sqrt{3x+1}+2)} \\
&= \lim_{x\to 1} \frac{3}{\sqrt{3x+1}+2} \\
&= \frac{3}{\sqrt{3+1}+2} = \frac{3}{4}.
\end{aligned}$$

**(c)** Find the equation of the tangent line (in slope-intercept form) to $f(x) = \sqrt{3x+1}$ at the point $x = 1$. (5 pts.)

**Answer:** $y = \frac{3}{4}x + \frac{5}{4}$. From part **(b)**, we have that $m = f'(1) = 3/4$. Thus, $y = \frac{3}{4}x + b$. To find $b$, we use the point $(1, 2)$ since $x = 1$ implies $y = f(1) = 2$. We have $2 = \frac{3}{4} \cdot 1 + b$, which implies $b = 2 - \frac{3}{4} = \frac{5}{4}$. Thus the equation of the tangent line is $y = \frac{3}{4}x + \frac{5}{4}$.

5. The graph below shows three functions: $f(x)$, $g(x)$, and $h(x)$. If $f'(x) = g(x)$ and $g'(x) = h(x)$, identify the graph that represents each function. Explain briefly how you arrived at your answer. (9 pts.)




**A.** $\underline{\quad h(x) \quad}$ **B.** $\underline{\quad g(x) \quad}$ **C.** $\underline{\quad f(x) \quad}$

**Answer:** Graph **A** (dashed curve) is the derivative of graph **B** (dotted curve). This can be seen from the fact that **B** is decreasing from 0 to 1, has zero slope at $x = 1$, and then increases from 1 to 3. Graph **A** is negative from 0 to 1, 0 at $x = 1$, and is positive from 1 to 3, so it must be the derivative of **B**.

Next, we have that graph **B** is the derivative of graph **C** (solid curve). This can be seen from the fact that **C** is decreasing from 0 to 2, has zero slope at $x = 2$, and then increases from 2 to 4. Graph **B** is negative from 0 to 2, 0 at $x = 2$, and is positive from 2 to 4, so it must be the derivative of **C**.

Putting both facts together, $f(x)$ is graph **C**, $g(x)$ is graph **B**, and $h(x)$ is graph **A**.

6. Some final conceptual questions. You must show your work to receive any partial credit. (24 pts.)

**(a)** If $6x - 4 \le h(x) \le x^2 + 5$ for all $x$, find $\lim_{x \to 3} h(x)$.

**Answer:** Using the Squeeze Theorem, since $\lim_{x \to 3} 6x - 4 = 14$ and $\lim_{x \to 3} x^2 + 5 = 14$, we have that $\lim_{x \to 3} h(x) = 14$.

**(b)** Find any horizontal asymptotes for the function $z(t) = \dfrac{7 - 3t^2 + 2\pi t^4}{9 + 4t - 5t^4}$.

**Answer:** $z = -2\pi/5$. To find horizontal asymptotes, we need to compute $\lim_{t \to \infty} z(t)$ and $\lim_{t \to -\infty} z(t)$. Since the highest power in the numerator and denominator is the same, namely $t^4$, we simply read off the coefficients of this term. Both limits give the same value of $-2\pi/5$.

**(c)** Find and simplify $F'(x)$ if $F(x) = \sqrt{x}\, e^x - \dfrac{2}{x^3} + \pi^2$.

**Answer:** $F'(x) = e^x(\frac{1}{2}x^{-1/2} + x^{1/2}) + 6/x^4$ or $\frac{e^x}{\sqrt{x}}(1/2 + x) + 6/x^4$.

We use the product rule and linearity. First, write $\sqrt{x}\, e^x$ as $x^{1/2}e^x$. By the product rule, the derivative of this term is

$$\frac{1}{2}x^{-1/2} \cdot e^x + x^{1/2} \cdot e^x \;=\; e^x \left( \frac{1}{2}x^{-1/2} + x^{1/2} \right).$$

Next, write $-\frac{2}{x^3}$ as $-2x^{-3}$. Using the power rule, the derivative of this term is $6x^{-4}$ or $6/x^4$. Finally, $\pi^2$ is just a constant, so its derivative is 0. Adding the two derivatives together (using linearity of the derivative) gives the answer.

**(d)** Suppose that $Q(x) = \dfrac{e^x}{g(x)}$ and that $g(2) = 3, g'(2) = -5$. Find $Q'(2)$ (give the **exact** answer, no decimals).

**Answer:** $8e^2/9$. By the quotient rule, we have

$$Q'(x) \;=\; \frac{g(x) \cdot e^x - e^x \cdot g'(x)}{(g(x))^2}.$$

Now plug in $x = 2$ and simplify. This yields

$$\begin{aligned} Q'(2) \;&=\; \frac{g(2) \cdot e^2 - e^2 \cdot g'(2)}{(g(2))^2} \\ &=\; \frac{3e^2 - e^2(-5)}{9} \;=\; \frac{8e^2}{9}. \end{aligned}$$