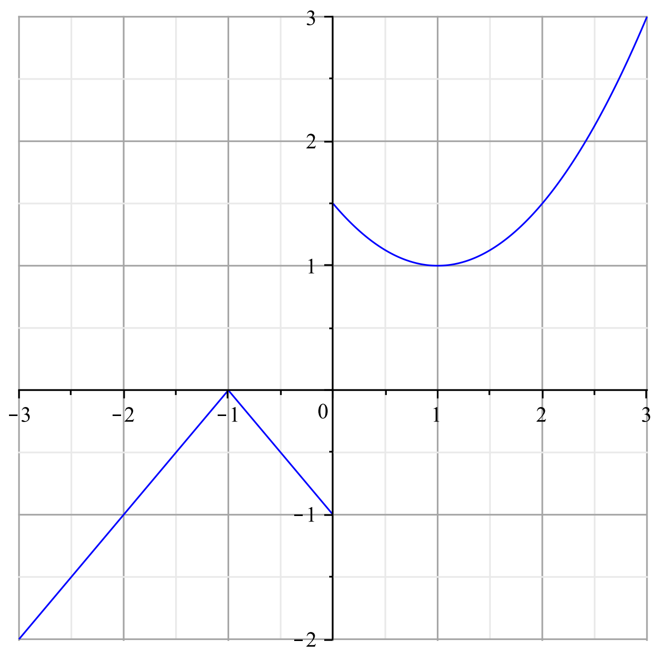
MATH 135-04 Calculus 1

Exam #1 SOLUTIONS **October 1, 2015** **Prof. G. Roberts**

1. The **ENTIRE** graph of $f(x)$ is shown below. Use it to answer each of the following questions: (18 pts.)



(a) What is the range of f?

Answer: $[-2, 0] \cup [1, 3]$. The range is found by projecting the graph onto the y-axis. Note that there is a gap in the function and that no y-values between 0 and 1 have pre-images in the domain.

(b) Is f a one-to-one function? Explain.

Answer: No, it fails the horizontal line test. Specifically, there are output y-values of the function that have more than one pre-image x in the domain. For example, $f(-1.5) = f(-0.5) = -0.5$.

(c) Suppose that $g(x) = \cos x$. Find the value of $(f \circ g)(9\pi)$.

Answer: 0.

$$(f \circ g)(9\pi) \ = \ f(g(9\pi)) \ = \ f(\cos(9\pi)) \ = \ f(-1) \ = \ 0.$$

(d) Evaluate each of the following limits:

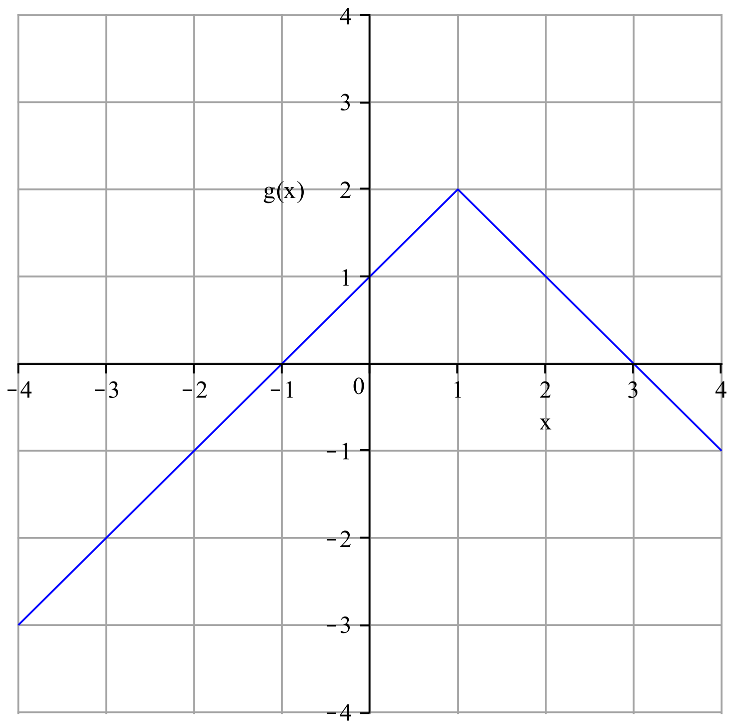
(i) $\lim_{x \to 0+} f(x) = 1.5$ **(ii)** $\lim_{x \to 0-} f(x) = -1$

(iii) $\lim_{x \to 0} f(x)$ does not exist because the left- and right-hand limits are not equal.

2. **The Absolute Value Function** (12 pts.)

(a) Carefully sketch the graph of $g(x) = -|x-1| + 2$ on the axes below.

Answer: Take the usual graph of the absolute value function (the V) and shift it to the right by one unit, reflect it over the x-axis, and then shift it up by 2 units. Plot a few points to get the slope of the lines correct.



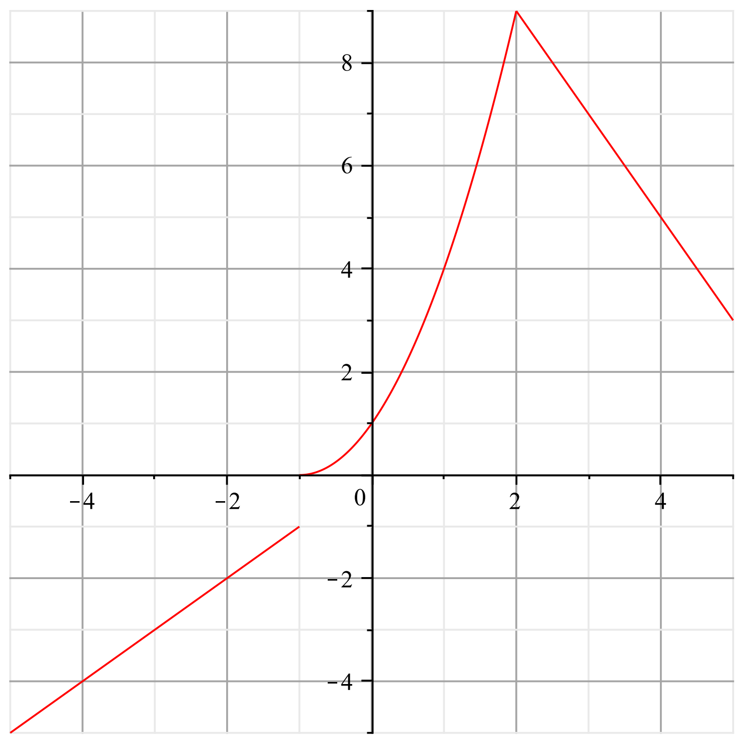
(b) Find all x satisfying $|3x+6| \geq 4$. You may express your answer in interval notation or using inequalities.

Answer: $(-\infty, -10/3] \cup [-2/3, \infty)$.
Based on the definition of the absolute value function, there are two different cases. First, if $3x+6 > 0$, then $|3x+6| = 3x+6$. Solving the inequality $3x+6 \geq 4$ leads to $x \geq -2/3$. The second case is when $3x+6 < 0$ (or when $x < -2$), which is a completely separate case from the first one (no overlap). Then we have $|3x+6| = -(3x+6) = -3x-6$. Solving the inequality $-3x-6 \geq 4$ gives $x \leq -10/3$. Thus there are two possible intervals that x could be in and satisfy the given inequality: $x \geq -2/3$ **OR** $x \leq -10/3$. In interval notation this is $(-\infty, -10/3] \cup [-2/3, \infty)$.

3. Carefully sketch the graph of the following piecewise function: (10 pts.)

$$f(x) = \begin{cases} x & \text{if } x < -1 \\ (x+1)^2 & \text{if } -1 \leq x < 2 \\ 13 - 2x & \text{if } 2 \leq x < 5 \end{cases}$$

Answer: Be sure to draw each part of the graph over the correct interval. The first piece is a line with slope 1; the second piece is a parabola opening up with vertex at $(-1, 0)$; and the final piece is another line, but with slope -2. The graph has a jump discontinuity at $x = -1$ but is actually continuous at $x = 2$ because the left- and right-hand limits are both equal to $f(2) = 9$.



4. **Trig is Fun** (20 pts.)

(a) The period of the function $g(x) = 4\sin(3\pi x)$ is 2/3.

Answer: Using the formula that the period of the function $y = \sin(bx)$ is $2\pi/b$, we substitute $b = 3\pi$ and simplify to obtain 2/3.

(b) State the domain and range of the function $h(x) = \cos^{-1}(x)$.

Domain: $[-1, 1]$ Range: $[0, \pi]$

Answer: The domain of $\cos^{-1}(x)$ is equal to the range of its inverse function, $\cos x$. Since the cosine of an angle is the x-coordinate on the unit circle, then it is always between -1 and 1. The range of $\cos^{-1}(x)$ is defined to be the angles between 0 and π. This interval is chosen in order to make the cosine function one-to-one (recall that a function must be one-to-one in order to have an inverse.)

(c) Find all angles θ between 0 and 2π that satisfy $\cos(\theta) = -1/2$. Give your answer(s) in radians.

Answer: $2\pi/3, 4\pi/3$. First, we find the reference angle β, so that $\cos(\beta) = 1/2$. Using a 30–60–90 right triangle, or from memory, $\beta = 60° = \pi/3$. Since $\cos(\theta) < 0$, we must chose θ to be in the second or third quadrant. Thus, the solution is $\theta = \pi - \pi/3 = 2\pi/3$ and $\theta = \pi + \pi/3 = 4\pi/3$.

(d) Suppose that $\tan\theta = -5/12$ and that $\pi/2 < \theta < \pi$. Find the values of $\sin\theta$ and $\sec\theta$.

Answer: $\sin\theta = 5/13$ and $\sec\theta = -13/12$.

Using SOH-CAH-TOA, draw a right triangle with sides 5 and 12, with the angle θ across from the side of length 5. By the Pythagorean Theorem, the hypotenuse has length 13. It follows that $\sin\theta = 5/13$ (opp./hyp.) and $\cos\theta = -12/13$ (adj./hyp.; negative because θ is in the second quadrant). Since $\sec\theta = 1/\cos\theta$, we have $\sec\theta = -13/12$.

Note: There was a typo on the exam which incorrectly stated that $\tan\theta = 5/12$ (missing negative sign). Because this was inconsistent with $\pi/2 < \theta < \pi$, no points were deducted for the answer $\sec\theta = 13/12$.

5. **Average and Instantaneous Velocity** (10 pts.)

(a) Suppose that $s(t) = 5t^2 - 3t$ represents the distance in feet a ball has traveled after t seconds. Compute the average velocity over the interval $[1, 4]$ (give the correct units).

Answer: 22 feet per second.

Using the formula average velocity is $(s(t_2) - s(t_1))/(t_2 - t_1)$, we compute the average velocity to be

$$\frac{s(4) - s(1)}{4 - 1} = \frac{(80 - 12) - (5 - 3)}{4 - 1} = \frac{66}{3} = 22 \text{ ft/sec.}$$

(b) Fill in the blanks:

If $s(t)$ represents the position function of a moving object, then the instantaneous velocity at the time $t = 3$ is defined as the <u>slope</u> of the <u>tangent line</u> to the graph of $s(t)$ at the point $t = 3$.

6. **Calculus Potpourri** (30 pts.)

(a) The function $g(x) = \cos x + x^2 - 3$ is <u>even</u>.

(odd, even, neither odd nor even, both odd and even)

Answer: Since $\cos x, x^2$ and -3 are all even functions (each graph is symmetric with respect to y-axis), the sum is also even. In other words, $g(-x) = g(x)$ and thus g is an even function.

(b) Simplify $\log_3(27) + \ln(e^{15})$.

Answer: 18. We have $\log_3(27) = 3$ since $3^3 = 27$. We also have $\ln(e^{15}) = 15$ since $\ln(x)$ and e^x are inverses. Thus, the answer is $3 + 15 = 18$.

(c) Find the equation of the line passing through the point $(-2, 3)$ and perpendicular to the line $6x - 3y = 2015$.

Answer: $y = (-1/2)x + 2$. First, we compute the slope of the line $6x - 3y = 2015$ by writing it in slope-intercept form: $y = 2x - 2015/3$. Thus, $m = 2$. The slope of a line perpendicular to this has slope $m_\perp = -1/2$. Therefore, our line has the form $y = (-1/2)x + b$. To find b, we substitute $x = -2$ and $y = 3$ into the previous equation to obtain $3 = (-1/2) \cdot -2 + b$, which gives $b = 2$.

(d) Complete the square to find the minimum value of the function $Q(x) = 2x^2 - 6x + 11$.

Answer: 13/2 or 6.5. To complete the square, we first factor out a 2, leaving the 11 outside the parentheses.

$$Q(x) \;=\; 2(x^2 - 3x + \underline{\hspace{1.5cm}}) + 11 + \underline{\hspace{1.5cm}}.$$

Next, we determine the constant to add inside the parentheses by taking half of -3 and squaring. This yields $9/4$. We add $9/4$ inside the parentheses which means that we are really adding $2 \cdot 9/4 = 9/2$ to the function. To balance this out, we subtract $9/2$ outside the parentheses:

$$Q(x) \;=\; 2\left(x^2 - 3x + \frac{9}{4}\right) + 11 - \frac{9}{2} \;=\; 2\left(x - \frac{3}{2}\right)^2 + \frac{13}{2}.$$

Since the graph of $Q(x)$ is a parabola opening up, the minimum value of the function is the y-coordinate of the vertex of the parabola. The vertex is $(3/2, 13/2)$ so the minimum value is $13/2$. Another way to see this is to observe that the term in the parentheses is zero only when $x = 3/2$. Thus, the minimum of the function occurs at $x = 3/2$ and is found as $Q(3/2) = 13/2$.

(e) Find the **exact** solution (no decimals) to the equation $\ln(5 + 2x) = \pi$.

Answer: The **exact** solution is $x = (e^{\pi} - 5)/2$. The first step is to raise both sides to the base e because e^x is the inverse of $\ln(x)$. This gives

$$e^{\ln(5+2x)} \;=\; e^{\pi} \quad \text{or} \quad 5 + 2x \;=\; e^{\pi}.$$

Next, subtract 5 from both sides and then divide by 2. There is no need (or use) for a calculator on this problem.

(f) Evaluate $\displaystyle\lim_{t\to 0} \frac{\sin(5t)}{4t}$.

Answer: 5/4 or 1.25. Using a calculator to evaluate the limit, plug in t-values very close, but **not** equal to 0. For instance plugging in $t = \pm 0.01$ into the function $\sin(5t)/4t$ gives 1.2494792, while plugging in $t = \pm 0.0001$ gives 1.249999948. Notice that the results are the same whether you plug in t-values to the left or right of 0. It appears that the limit is $1.25 = 5/4$, a fact that we will soon learn to prove rigorously.