MATH 135 Calculus 1, Fall 2015
Worksheet for Sections 1.4 and 1.5

1.4 Trigonometric Functions

Radians

Angles in trigonometry are measured in radians, which corresponds to an actual physical length (as opposed to degrees, which is based on the fact that 360 has many factors).

Definition 0.1 *An angle of 1 radian is equal to the angle made by 1 unit of arc length along the unit circle.*

To convert between radians and degrees, use the formula

$$180° \ = \ \pi \text{ radians.} \tag{1}$$

Exercise 0.2 *Convert each of the following from radians to degrees or vice-versa.*

a. $270° = $ _____ **b.** $-135° = $ _____ **c.** $5\pi/6 = $ _____ **d.** $-4\pi/3 = $ _____

The Sine and Cosine Functions

There are multiple definitions of the sine function $y = \sin\theta$, but one of the simplest to understand is that it represents the y-coordinate on the unit circle at an angle of θ radians.

Definition 0.3 *The sine of θ, denoted by $\sin(\theta)$ or just $\sin\theta$, is the y-coordinate of the point of intersection between the unit circle and a ray emanating from the origin at an angle of θ radians. The cosine of θ, denoted by $\cos(\theta)$ or just $\cos\theta$, is the x-coordinate.*

It is important to remember that the input into the functions $f(\theta) = \sin\theta$ or $g(\theta) = \cos\theta$ is an *angle* θ. Since the unit circle has a radius of one, the x- and y-coordinates of any point on the unit circle always lie between -1 and 1. Thus, the range of $\sin\theta$ and $\cos\theta$ is $[-1, 1]$. By the Pythagorean Theorem, we have a fundamental relationship between $\sin\theta$ and $\cos\theta$:

$$\cos^2(\theta) + \sin^2(\theta) \ = \ 1 \quad \text{for any angle } \theta. \tag{2}$$

Another definition for the sine or cosine of an angle comes from using right triangles. This only works for finding the trig function of an angle between 0 and $\pi/2$. Recall the mnemonic phrase SOH-CAH-TOA, which reminds us of the following definitions:

$$\sin\theta \ = \ \frac{\text{opp.}}{\text{hyp.}}, \qquad \cos\theta \ = \ \frac{\text{adj.}}{\text{hyp.}}, \qquad \tan\theta \ = \ \frac{\text{opp.}}{\text{adj.}}.$$

Exercise 0.4 *Without using a calculator, evaluate each of the following:*

a. $\sin(3\pi/2) = $ _____ **b.** $\cos(45\pi) = $ _____ **c.** $\sin(7\pi/6) = $ _____ **d.** $\tan(3\pi/4) = $ _____

Other Trig Functions

There are four other trigonometric functions, each of which can be defined in terms of $\sin\theta$ and $\cos\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}. \tag{3}$$

Exercise 0.5 *Use the fundamental identity given in equation (2) to derive the identities*

$$\tan^2\theta + 1 = \sec^2\theta \qquad and \qquad \cot^2\theta + 1 = \csc^2\theta.$$

Exercise 0.6 *Suppose that $\sin\theta = \dfrac{3}{5}$. Find the value of the other five trigonometric functions.*

Exercise 0.7 *Find all solutions to the equation $\sin\beta = -\dfrac{1}{2}$ assuming that $0 \leq \beta \leq 2\pi$.*

Exercise 0.8 *Sketch a graph of the following trig functions. State the amplitude and period in each case.*

a. $y = 2\sin(3\theta)$

b. $y = -3\cos\left(\dfrac{\theta}{2}\right)$

1.5 Inverse Functions

One of the simplest ways to obtain a new function from an old one is to simply flip the domain and range. So if $f(2) = 7$, then the new function f^{-1}, called the **inverse of** f, has $f^{-1}(7) = 2$. The function f^{-1} simply maps each element in the range of f back to the element in the domain it came from; it "inverts" f.

However, there will be a problem if an element b in the range has two or more elements in the domain, say a_1 and a_2, that map to it (i.e., $f(a_1) = f(a_2) = b$). Then, the inverse of b would not be unique (it could be either a_1 or a_2), and f^{-1} would not be a function. To rectify this, we must assume that f is **one-to-one**, that is, each element in the range of f has one and only one pre-image that was sent to it in the domain.

For example, the function $f(x) = x^2$ is not one-to-one because both -2 and 2 are each sent to the same element in the range, $f(-2) = f(2) = 4$. This function fails the **horizontal line test** and is really two-to-one. However, if we restrict the domain (yes, it's ok to do this) to $x \geq 0$, then f suddenly becomes one-to-one and now the inverse is actually a function. You know it already as $f^{-1}(x) = \sqrt{x}$. By definition (i.e., restricting the domain of x^2), $\sqrt{x}$ only spits out non-negative values.

Key Point: The only functions with well-defined inverses are those that are **one-to-one**. They must pass the **horizontal line test.**

Note that the notation for the inverse of f is not the usual exponent notation. In other words,

$$f^{-1} \neq \frac{1}{f}$$

The choice of -1 as the exponent is mathematical shorthand for **inverse**. Don't confuse this!

Based on the definition of the inverse of a function, the following formulas should make sense:

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x.$$

Simply put, the inverse of f reverses what f does to x. Likewise, f reverses the action of f^{-1}, that is, the inverse of f^{-1} is just f.

Graphically, if (x, y) is a point on the graph of f, then (y, x) is a point on the graph of f^{-1}. Thus, to obtain the graph of f^{-1} from the graph of f (or vice versa), just reflect the graph of f about the line $y = x$. This is also how one obtains an analytic formula for the inverse of a function: interchange the variables x and y and solve for y.

Exercise 0.9 *Find the inverse of the function* $f(x) = 1 + \sqrt{2 + 3x}$. *What should the domain of* f^{-1} *be in order to have* f *as its inverse?*

Inverse Trig Functions

The key to defining the inverse trig functions is to restrict the domains of the original trig functions in order to ensure that they are one-to-one. For example, the sine function is one-to-one on the domain $-\pi/2 \le \theta \le \pi/2$ (check the graph). By making this restriction, we then **define** the range of the inverse sine function (also called the **arcsine function**) to be $[-\pi/2, \pi/2]$. The domain of the inverse sine function is $[-1, 1]$ because this is precisely the range of the sine function.

Key Point: The inverse sine function, denoted $\sin^{-1}(x)$, inputs numbers between -1 and 1 and outputs angles between $-\pi/2$ and $\pi/2$. If $\theta = \sin^{-1}(x)$, then $\sin(\theta) = x$. (Go backwards!)

The outputs of the other inverse trig functions are given below:

- Range of $\cos^{-1}(x)$ is $[0, \pi]$.

- Range of $\sec^{-1}(x)$ is $[0, \pi]$ (except for $\pi/2$).

- Range of $\tan^{-1}(x)$ is $(-\pi/2, \pi/2)$.

- Range of $\cot^{-1}(x)$ is $(0, \pi)$.

- Range of $\csc^{-1}(x)$ is $[-\pi/2, \pi/2]$ (except for 0).

Exercise 0.10 *Evaluate each of the following without using a calculator:*

a. $\sin^{-1}(1/\sqrt{2}) = $ _____

b. $\sin^{-1}(1) = $ _____

c. $\cos^{-1}(-\sqrt{3}/2) = $ _____

d. $\tan^{-1}(-1) = $ _____

e. $\sec^{-1}(2) = $ _____

f. $\csc^{-1}(-1) = $ _____

Exercise 0.11 *Explain why $\pi/2$ must be excluded from the range of $\sec^{-1}(x)$. In other words, why does the equation $\sec^{-1}(x) = \pi/2$ have no solution?*

Exercise 0.12 *Explain why $\cos^{-1}(\cos(17\pi)) = \pi$ and not 17π.*