MATH 135 Calculus 1, Fall 2015

Worksheet for Section 3.10: Related Rates

Key Idea: If two quantities are related by some equation and **both** are changing with respect to time, then we can use the given equation and the chain rule to "relate their rates." For example, suppose we have two quantities $x \equiv x(t)$ and $y \equiv y(t)$, which are related to each other by the equation

$$2\sin(x) \ = \ y^2 - y - 1 \quad \text{or} \quad 2\sin(x(t)) \ = \ (y(t))^2 - y(t) - 1.$$

Recall that dx/dt is the rate of change of x with respect to time t, and dy/dt is the rate of change of y with respect to time t. To obtain a relation between dx/dt and dy/dt, we take the derivative of the above equation with respect to t and use the chain rule. This gives

$$2\cos(x) \cdot \frac{dx}{dt} \ = \ 2y \cdot \frac{dy}{dt} - \frac{dy}{dt}. \tag{1}$$

Check equation (1) carefully to be sure you understand how the chain rule works here. It is similar to the techniques we use for implicit differentiation. Note that there are four unknown quantities in equation (1): x, y, dx/dt and dy/dt. If we know the values for three of these quantities, we can plug into equation (1) to find the remaining quantity.

Some Useful Geometric Formulas: Many of the applications using related rates involve the area or volume of some common geometric objects. Some good formulas to know include:

$$\begin{gathered} \text{area of a circle} \quad A = \pi r^2 \\ \text{volume of a sphere} \quad V = \tfrac{4}{3}\pi r^3 \\ \text{volume of a cylinder} \quad V = \pi r^2 h \\ \text{volume of a right-circular cone} \quad V = \tfrac{1}{3}\pi r^2 h \end{gathered}$$

Example 1: (A Warm-up Problem) Suppose that each side of a square is increasing at a constant rate of 10 cm/sec. At what rate is the area increasing when the area of the square is 36 cm^2?

Partial Solution: First, let's define our variables. Let x be the common side length of the square, and let A be the area of the square. We are given that $dx/dt = 10$ and are asked to find dA/dt when $A = 36$. (Use the units given in the problem to help determine what each quantity represents, e.g., cm/sec indicates a rate of change.) To finish the problem, write down an equation relating A and x, and then use the chain rule to derive a simple relation between dx/dt and dA/dt. Plug the given information into your relation and solve for dA/dt. You should get an answer of 120 cm^2/sec.

Example 2: (Rock in a Pond) A rock is thrown into a pond and the resulting splash produces a circle whose radius increases at a constant rate of 40 cm/sec. How fast is the area of the circle increasing when the radius is 25 cm? Give the **exact** answer.

Example 3: (The Sliding Ladder Problem) A ladder 5 feet in length is resting against a wall when suddenly, it begins to slide vertically down the wall. If the bottom of the ladder slides away from the wall at a constant rate of 0.5 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 feet away from the wall?

Example 4: (Conical Tank) A tank in the shape of an upright circular cone (vertex on the bottom) is being filled with water at a rate of 6 m^3/sec. The height of the tank is 5 meters and the radius is 3 meters. How fast is the water level rising when the water level is 2 meters? How fast is the water level rising when the tank is half full? *Hint:* Use similar triangles.

Example 5: (Tracking a Rocket) A spy uses a telescope to track a rocket that has been launched vertically from a launch pad 4 miles away. When the angle between the telescope and the ground is equal to $\pi/4$, the angle is changing at a rate of 1.2 radians per minute. How fast is the rocket traveling at this particular instant?