MATH 135 Calculus 1, Fall 2015

Worksheet for Sections 3.2 and 3.3

3.2 The Derivative as a Function

Recall that the derivative of a function at a point gives the slope of the tangent line at that point. In other words, $f'(a)$ represents the slope of the tangent line at $x = a$. In this section, we vary the point a, and treat the derivative as a function in its own right, the function $f'(x)$. The definition is the same as before, except that now we replace a by the variable x.

Definition 0.1 *The* **derivative function** $f'(x)$ *is given by*

$$\boxed{f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\,.} \tag{1}$$

The derivative function inputs the number x and outputs the slope of the tangent line to f at x. Recall that $f'(x)$ does not exist if f has a corner, cusp, or vertical tangent line at x.

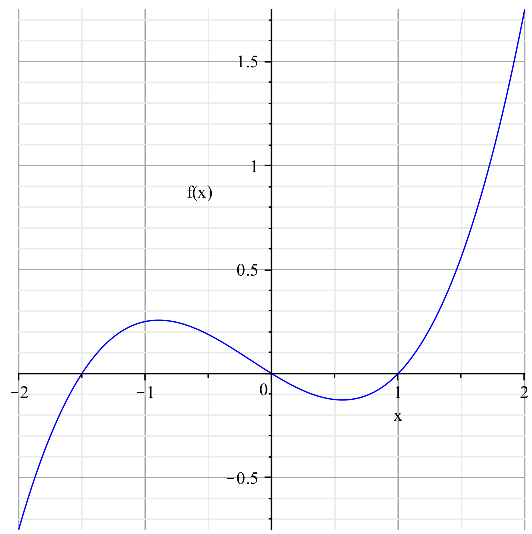
In class we used the limit definition to derive a few key derivative formulas. The symbol $\frac{d}{dx}$ (Leibniz notation) means to take the derivative with respect to x.

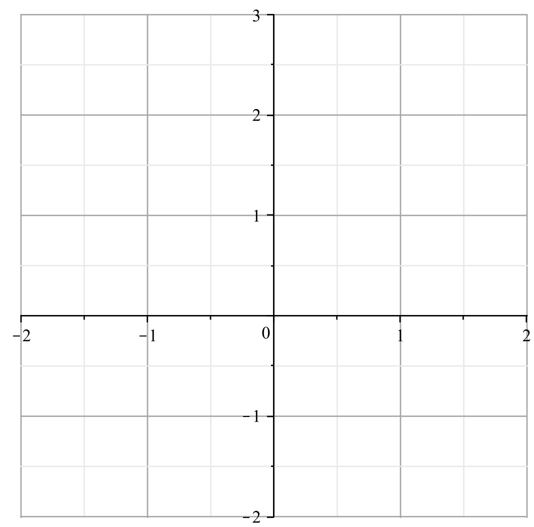
1. $\frac{d}{dx}(c) \;=\; 0$ (The derivative of a constant is zero.)
2. $\frac{d}{dx}(mx+b) \;=\; m$ (The derivative of a line is its slope.)
3. $\frac{d}{dx}(x^n) \;=\; nx^{n-1}$ for **any** real number n. (Power Rule)
4. $\frac{d}{dx}(cf(x)) \;=\; cf'(x)$ (Constants pull out.)
5. $\frac{d}{dx}(f(x)+g(x)) \;=\; f'(x)+g'(x)$ and $\frac{d}{dx}(f(x)-g(x)) \;=\; f'(x)-g'(x)$ (Linearity)
6. $\frac{d}{dx}(e^x) \;=\; e^x$ (The derivative of e^x is itself.)
7. $\frac{d}{dx}(b^x) \;=\; (\ln b)\cdot b^x$ (The derivative of an exponential function is a constant times itself.)

Exercise 0.2 *Find the derivative of the function* $g(x) = 6\sqrt{x} - \frac{3}{x^3} + 5e^x - \pi^4$.

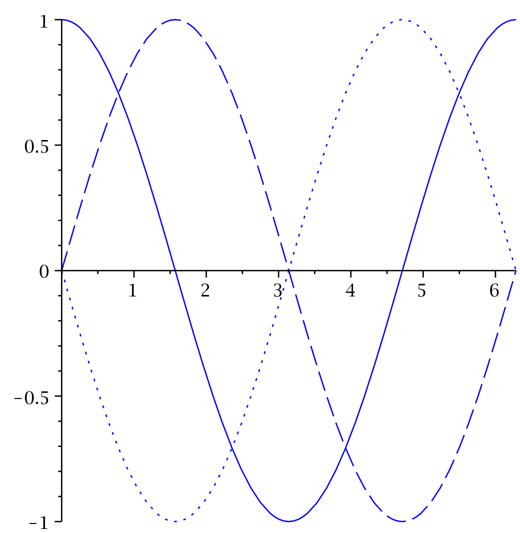
Exercise 0.3 *Using Equation (1), explain why* $\dfrac{d}{dx}\left(e^{x}\right) = e^{x}$.

Exercise 0.4 *Given the graph of* $f(x)$ *below, sketch the graph of the derivative function* $f'(x)$ *on the adjacent plot.* **Hint:** *Input* x*, output slope. Focus on the sign of the derivative first.*





Exercise 0.5 *The graph below shows three functions:* $f(x), g(x),$ *and* $h(x)$*. If* $f'(x) = g(x)$ *and* $g'(x) = h(x)$*, identify the graph that represents each function. Explain.*



Exercise 0.6 *If the graph of $g(t)$ is a parabola, what type of graph will $g'(t)$ be? Explain.*

Exercise 0.7 *Find* $\frac{d}{dx}(e^x + x^e)$.

3.3 Product and Quotient Rules

There are two useful rules for computing the derivative of a product and quotient of two functions. Interestingly, Leibniz himself messed up the product rule in an early draft of his manuscript on calculus.

Theorem 0.8 (Product Rule) *If $f(x)$ and $g(x)$ are differentiable functions, then so is their product $f(x) \cdot g(x)$. The derivative of the product is given by*

$$\boxed{\frac{d}{dx}\left(f(x) \cdot g(x)\right) \;=\; f'(x)g(x) + f(x)g'(x)\,.} \tag{2}$$

Notice the symmetry in Formula (2) and that it is **not** the case that the derivative of the product equals the product of the derivatives. For instance, suppose we applied the Product Rule to take the derivative of $x \cdot x$. If we just multiplied the product of the derivatives, we would get

$$\frac{d}{dx}(x \cdot x) \;=\; \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) \;=\; 1 \cdot 1 \;=\; 1\,???$$

But this is clearly incorrect since $x \cdot x = x^2$ and the derivative of x^2 is $2x$ by the Power Rule. A correct application of the Product Rule is as follows:

$$\frac{d}{dx}(x \cdot x) \;=\; \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) \;=\; 1 \cdot x + x \cdot 1 \;=\; 2x.$$

Exercise 0.9 *Use the Product Rule to find $f'(x)$ where $f(x) = (3x^2 + 1)e^x$. Simplify your answer.*

Theorem 0.10 (Quotient Rule) *If $f(x)$ and $g(x)$ are differentiable functions, then so is their quotient $f(x)/g(x)$ as long as $g(x) \neq 0$. The derivative of the quotient is given by*

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,.} \tag{3}$$

Exercise 0.11 *Derive the Quotient Rule from the Product Rule. Start by letting $Q(x) = \dfrac{f(x)}{g(x)}$. Cross multiply, use the Product Rule, and then solve for $Q'(x)$.*

Exercise 0.12 *Use the Quotient Rule to calculate the derivative of $\dfrac{1}{x^4}$ and check your answer against the result obtained from the Power Rule.*

Exercise 0.13 *If $g(x) = \dfrac{3x+1}{2x-5}$, find and simplify $g'(x)$.*

Exercise 0.14 *If $h(x) = \dfrac{e^x}{x^2+1}$, find and simplify $h'(x)$.*