MATH 135 Calculus 1, Fall 2015

Worksheet for Sections 2.4 and 2.5

2.4 Limits and Continuity

Continuity

Intuitively, a continuous function is one that can be drawn without having to lift up your pencil. Functions that are *not* continuous have holes, jumps, asymptotes, or places where limits don't exist (e.g., infinite oscillations). The precise mathematical definition involves limits.

Definition 0.1 *A function $f(x)$ is continuous at $x = a$ if*

$$\boxed{\lim_{x \to a} f(x) \; = \; f(a).} \tag{1}$$

For a function to be continuous at the point $x = a$, there are three conditions:

1. $f(a)$ must exist (there must be a function value).
2. The limit of the function as x approaches a must exist, and it must equal a real number (so ∞ or $-\infty$ is not ok).
3. The limit must equal the function value.

Types of Discontinuities (see class notes from 9/24 for graphical examples)

- If the left-hand and right-hand limits at $x = a$ each exist, but are not equal to each other, then f has a **jump discontinuity** at $x = a$.
- If $\lim_{x \to a} f(x)$ exists, but does not equal the function value $f(a)$, then f has a **removable discontinuity** at $x = a$. If we redefine $f(a)$ to be the value of the limit, then the function becomes continuous at $x = a$, and we have "removed" the discontinuity.
- If either of the one sided limits go to ∞ or $-\infty$, then f has an **infinite discontinuity** at $x = a$.

One-Sided Continuity

A function is **left continuous** at $x = a$ if the left-hand limit exists and equals the function value $f(a)$. Similarly, it is **right continuous** at $x = a$ if the right-hand limit exists and equals the function value $f(a)$. In other words,

- The function $f(x)$ is left continuous at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.
- The function $f(x)$ is right continuous at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

Exercise 0.2 *Consider the piecewise function $f(x)$ defined as follows:*

$$f(x) = \begin{cases} 4 & \text{if } x \le 1 \\ 4-x & \text{if } 1 < x \le 4 \\ (x-4)^2 & \text{if } x > 4. \end{cases}$$

(i) *Carefully sketch the graph of $f(x)$.*

(ii) *Describe the type of continuity (if any) at $x = 1$.*

(iii) *Describe the type of continuity (if any) at $x = 4$.*

(iv) *How would you change the definition of the function over $1 < x \le 4$ to make it continuous for all real numbers?*

Note: Polynomials, rational functions, exponentials, logs, and trig functions are all continuous on their domains. Moreover, the composition of continuous functions is also continuous. For example, the function

$$g(x) = \sin\left(e^{x^2-1}\right)$$

is continuous since it is the composition of the continuous functions $x^2 - 1$, e^x and $\sin x$. To find limits of continuous functions, we simply evaluate the function at the point in question (i.e, just plug it in!) Thus, since g is continuous at $x = 1$, we have

$$\lim_{x \to 1} g(x) = g(1) = \sin(e^{1-1}) = \sin(1).$$

Exercise 0.3 *Use continuity to find the value of each limit.*

(i) $\lim_{t \to 3} \log_5 [\cos(t-3) + 4]$

(ii) $\lim_{x \to -1} \dfrac{3^x}{\sqrt{x+5}}.$

2.5 Evaluating Limits Algebraically

If a function is continuous at a point, then the limit of the function is easily found by simple substitution. For instance, $\lim_{x\to 4} x^2 - 3 = 4^2 - 3 = 13$. Otherwise, we may use a graph to try and discern the limit or perform numerical calculations (plug in values really, really close to the point $x = a$) to find the limit of the function as x approaches a.

This section is concerned with a new technique, namely, using algebra to calculate limits. Here is a simple example. Consider the problem

$$\lim_{x\to 3} \frac{x-3}{x^2-9}\ .$$

The function is clearly not continuous at $x = 3$ because the denominator becomes 0 when $x = 3$. However, notice that the numerator is also 0 when $x = 3$. Thus, our limit has the form $\frac{0}{0}$, which is called an **indeterminate form**. In this case, there is often some algebra (e.g., factoring) that can be performed to simplify the function and compute the limit by hand.

We have

$$\lim_{x\to 3} \frac{x-3}{x^2-9} = \lim_{x\to 3} \frac{x-3}{(x-3)\cdot(x+3)} = \lim_{x\to 3} \frac{1}{x+3} = \frac{1}{6}.$$

The cancellation is valid here because we never reach $x = 3$ in the limit, so that $x - 3 \neq 0$ and can be cancelled from the top and bottom of the fraction.

Here is a list of indeterminate forms. A limit that takes one of these forms can be *anything* (hence the name indeterminate), so further work must be done to find the actual value of the limit.

$$\boxed{\textbf{Key indeterminate forms: } \frac{0}{0},\ \frac{\infty}{\infty},\ \infty\cdot 0,\ \infty-\infty.} \tag{2}$$

Exercise 0.4 *Find the value of the limit by first canceling a common factor from the numerator and denominator.*

$$\lim_{x\to 2} \frac{x^2-4}{x^2+4x-12}$$

Exercise 0.5 *If $f(x) = 5x^2 - 3x$, find the value of $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$. Limits of this form are very important in Calculus.*

Exercise 0.6 *Find the value of* $\lim_{x\to 9} \frac{\sqrt{x}-3}{9-x}$. **Hint:** *Multiply top and bottom by the conjugate* $\sqrt{x}+3$.

Exercise 0.7 *Find the value of* $\lim_{\theta\to\pi/2} \frac{\tan\theta}{\sec\theta}$.

Exercise 0.8 *Find the value of each one-sided limit.* **Hint:** *Use the definition of the absolute value function.*

(i) $\lim_{x\to 3^-} \frac{|4x-12|}{x-3}$

(ii) $\lim_{x\to 3^+} \frac{|4x-12|}{x-3}$

Exercise 0.9 *Find the value of* $\lim_{x\to 1} \frac{6}{x^2-1} - \frac{3}{x-1}$. **Hint:** *Add the fractions and simplify.*