MATH 135 Quiz #3 Solutions

September 20, 2013 **Prof. G. Roberts**

1. Consider the parametric equations $x = \sin(2\theta)$, $y = \cos(2\theta)$, $0 \leq \theta \leq \pi$.

   (a) Find a Cartesian equation for the curve using only x and y variables. (2 pts.)

   Answer: $x^2 + y^2 = 1$.

   Recall the fundamental identity $\cos^2 t + \sin^2 t = 1$, which holds for *any* angle t. In particular, it holds when $t = 2\theta$. Thus, $\sin^2(2\theta) + \cos^2(2\theta) = 1$, which becomes $x^2 + y^2 = 1$ upon substitution.

   (b) Fill in the blanks: The curve traced out by the given parametric equations is a

   circle . It begins at the point $(0, 1)$ and ends at the point $(0, 1)$,

   and is traversed in the clockwise direction. (4 pts.)

   Answer: We know the curve traced out is a circle because the Cartesian equation is $x^2+y^2 = 1$, the equation of the unit circle. To find where it starts and ends, plug in $\theta = 0$ and $\theta = \pi$, respectively, into the equations for x and y. At $\theta = 0$, we have $x = \sin(2 \cdot 0) = \sin(0) = 0$ and $y = \cos(2 \cdot 0) = \cos(0) = 1$, so the curve begins at the point $(0, 1)$. At $\theta = \pi$, we have $x = \sin(2 \cdot \pi) = \sin(2\pi) = 0$ and $y = \cos(2 \cdot \pi) = \cos(2\pi) = 1$, so the curve ends at the point $(0, 1)$ as well.

   The fact that the curve starts and ends at the same point suggests that the full circle is traced out. This is correct. Even though the parameter θ ranges from only 0 to $\pi = 180°$, 2θ ranges from 0 to 2π, so the entire circle is traced out.

   Finally, the circle is traced out clockwise because both x and y are positive for values of the parameter slightly greater than 0. It follows that the curve starts at $(0, 1)$ and then moves into the first quadrant (clockwise direction). Alternatively, one could plug in a good value like $\theta = \pi/4$, and see that at $\theta = \pi/4$, $x = \sin(2 \cdot \pi/4) = \sin(\pi/2) = 1$ and $y = \cos(2 \cdot \pi/4) = \cos(\pi/2) = 0$, so the curve is passing through the point $(1, 0)$ one-quarter of the way through its motion.

2. The position of a particle moving along a straight line is given by $s(t) = 3\sin t - 2\cos t$. Find the average velocity over the time interval $[\pi/2, \pi]$. Be sure to simplify your answer. (4 pts.)

   Answer: $-2/\pi$

   Using the formula for average velocity over the interval $[t_1, t_2]$, $\dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$, we have

$$\begin{aligned}
\frac{s(\pi) - s(\pi/2)}{\pi - \pi/2} &= \frac{(3\sin(\pi) - 2\cos(\pi)) - (3\sin(\pi/2) - 2\cos(\pi/2))}{\pi - \pi/2} \\
&= \frac{(3 \cdot 0 - 2 \cdot (-1)) - (3 \cdot 1 - 2 \cdot 0)}{\pi/2} = \frac{2 - 3}{\pi/2} \\
&= \frac{-1}{\pi/2} \\
&= \frac{-1}{1} \cdot \frac{2}{\pi} = -\frac{2}{\pi}.
\end{aligned}$$