MATH 135 Calculus 1

Exam #3 Solutions **November 25, 2013** **Prof. G. Roberts**

1. Calculate the derivative of each function. **SIMPLIFY** your answer as best as possible. (30 pts.)

(a) $f(x) = \dfrac{5}{x^5} + 5^x + e^5$

Answer: First, to avoid using the quotient rule, we write $f(x) = 5x^{-5} + 5^x + e^5$. Then, using the power rule and Rule 5 for differentiating an exponential function, we have

$$f'(x) \;=\; -25x^{-6} + 5^x \cdot \ln 5 \;=\; \frac{-25}{x^6} + (\ln 5) \cdot 5^x.$$

Note that e^5 is just a constant, so its derivative is 0.

(b) $g(x) = \sqrt{\tan(x^2+1)}$

Answer: First write $g(x) = (\tan(x^2+1))^{1/2}$. Applying the chain rule twice, we have

$$g'(x) \;=\; \frac{1}{2}(\tan(x^2+1))^{-1/2} \cdot \sec^2(x^2+1) \cdot 2x \;=\; \frac{x\sec^2(x^2+1)}{\sqrt{\tan(x^2+1)}}$$

(c) $h(t) = e^{\tan^{-1}(\sqrt{t})}$

Answer:

$$h'(t) \;=\; e^{\tan^{-1}(\sqrt{t})} \cdot \frac{1}{1+(\sqrt{t})^2} \cdot \frac{1}{2}t^{-1/2} \;=\; \frac{e^{\tan^{-1}(\sqrt{t})}}{2\sqrt{t}\,(1+t)}$$

(d) $G(\theta) = \ln(\,\ln(\cos\theta)\,)$

Answer:

$$G'(\theta) \;=\; \frac{1}{\ln(\cos\theta)} \cdot \frac{1}{\cos(\theta)} \cdot -\sin(\theta) \;=\; \frac{-\tan\theta}{\ln(\cos\theta)}$$

(e) $y = x^{\arcsin(3x)}$

Answer: Using logarithmic differentiation, we take the natural log of both sides to obtain

$$\ln y \;=\; \ln x^{\arcsin(3x)} \;=\; \arcsin(3x) \cdot \ln x.$$

Next we differentiate both sides with respect to x, using implicit differentiation on the left and the product rule on the right. This gives

$$\frac{1}{y} \cdot \frac{dy}{dx} \;=\; \frac{1}{\sqrt{1-(3x)^2}} \cdot 3 \cdot \ln x + \arcsin(3x) \cdot \frac{1}{x} \;=\; \frac{3\ln x}{\sqrt{1-9x^2}} + \frac{\arcsin(3x)}{x}.$$

Multiplying both sides by y and substituting in for y gives the answer

$$\frac{dy}{dx} \;=\; x^{\arcsin(3x)}\left(\frac{3\ln x}{\sqrt{1-9x^2}} + \frac{\arcsin(3x)}{x}\right).$$

2. For the equation below, use implicit differentiation to calculate dy/dx. (12 pts.)

$$x^3 \sin y + \cos(2y) = e^{y^3} + 7x^2$$

Answer: Differentiating each side with respect to x and treating $y = y(x)$ as a function of x, we have, by the chain rule,

$$3x^2 \sin y + x^3 \cos y \cdot \frac{dy}{dx} - \sin(2y) \cdot 2\frac{dy}{dx} \;=\; e^{y^3} \cdot 3y^2 \cdot \frac{dy}{dx} + 14x.$$

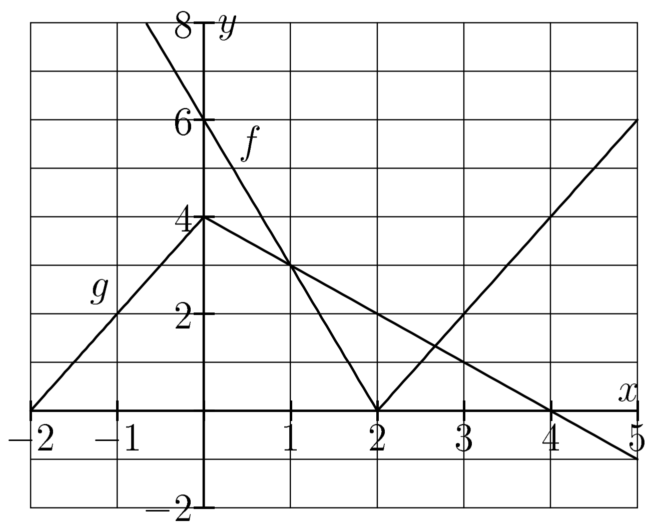
Grouping all terms with dy/dx together on one side of the equation yields

$$x^3 \cos y \cdot \frac{dy}{dx} - 2\sin(2y) \cdot \frac{dy}{dx} - 3y^2 e^{y^3} \cdot \frac{dy}{dx} \;=\; 14x - 3x^2 \sin y$$

which gives, after factoring out the dy/dx on the left-hand side,

$$\frac{dy}{dx} \;=\; \frac{14x - 3x^2 \sin y}{x^3 \cos y - 2\sin(2y) - 3y^2 e^{y^3}}.$$

3. The graphs of the functions $f(x)$ and $g(x)$ are given below. Suppose that $Q(x) = f(x)/g(x)$ and $C(x) = f(g(x))$. If they exist, find both $Q'(3)$ and $C'(4)$. (12 pts.)



Answer: $Q'(3) = 4$ and $C'(4) = 3$.

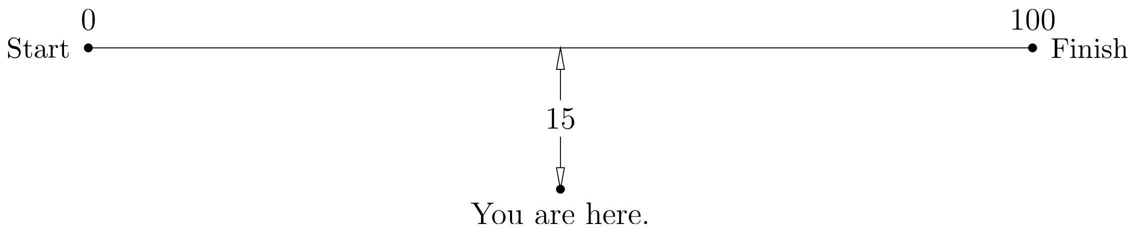
To find $Q'(3)$, use the quotient rule and fill in the appropriate values according to the graph (derivative = slope). We have that $Q'(x) = (g(x)f'(x) - f(x)g'(x))/(g(x))^2$ so that

$$\begin{aligned} Q'(3) &= \frac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2} \\ &= \frac{1 \cdot 2 - 2 \cdot (-1)}{1^2} \\ &= 4. \end{aligned}$$

To find $C'(4)$, use the chain rule. We have that $C'(x) = f'(g(x)) \cdot g'(x)$ so that

$$C'(4) \;=\; f'(g(4)) \cdot g'(4) \;=\; f'(0) \cdot g'(4) \;=\; -3 \cdot -1 \;=\; 3.$$

4. You are at a Holy Cross track meet watching the 100 meter dash, sitting in the bleachers 15 meters back from the midway point of the race. Your friend Sylvie (a.k.a., the speedster) is running down the track at a constant speed of 9 meters per second. How fast is the distance between you and your friend changing when she is 75 meters into the race? Round your answer to two decimal places (12 pts.)



Answer: 7.72 meters/sec

Draw a right triangle with one leg equal to 15, the other leg equal to x and the hypotenuse equal to z. The right angle of this triangle occurs at the midway point of the race, 50 meters along the track. The hypotenuse z is the distance between you and Sylvie. Both the quantities x and z are changing over time, but the distance 15 is fixed because you are not moving. When Sylvie is 75 meters into the race, we have $x = 25$. We want to find dz/dt when $x = 25$.

By the Pythagorean Theorem, we have $x^2 + 15^2 = z^2$. Differentiating this equation with respect to t yields $2x \cdot \frac{dx}{dt} = 2z \cdot \frac{dz}{dt}$ which simplifies to

$$\frac{dz}{dt} \;=\; \frac{x}{z} \cdot \frac{dx}{dt}.$$

We are given that $dx/dt = 9$. To find z when $x = 25$, use the Pythagorean Theorem to obtain $z = \sqrt{25^2 + 15^2} = \sqrt{850} = 5\sqrt{34} \approx 29.155$. Therefore, we find that

$$\frac{dz}{dt} \;=\; \frac{25}{5\sqrt{34}} \cdot 9 \;=\; \frac{45}{\sqrt{34}} \;\approx\; 7.72 \text{ meters/sec.}$$

5. Find the absolute maximum and absolute minimum values of the function $f(x) = \frac{1}{2}x - \ln(2x)$ over the interval $[\frac{1}{2}, 6]$. (12 pts.)

Answer:

First, we find any critical points of f on the given interval. Using the chain rule, we have

$$f'(x) \;=\; \frac{1}{2} - \frac{1}{2x} \cdot 2 \;=\; \frac{1}{2} - \frac{1}{x}.$$

Solving $f'(x) = 0$ for x gives $1/2 = 1/x$ or $x = 2$. Next, compute the values of f at the endpoints of the given interval. We have $f(1/2) = 1/4 - \ln 1 = 1/4$ and $f(6) = 3 - \ln(12) \approx$

0.515. Since $f(2) = 1 - \ln 4 \approx -0.386$, we find (comparing the three function values) that f has an absolute maximum of $3 - \ln(12) \approx 0.515$ at $x = 6$ and an absolute minimum of $1 - \ln 4 \approx -0.386$ at $x = 2$.

6. Some final conceptual questions. You must **show your work** to receive any partial credit.

(a) State the domain and range of $f(x) = \cos^{-1}(x)$. (4 pts.)

Answer: Domain: $\underline{[-1, 1]}$ Range: $\underline{[0, \pi]}$
Remember that the output of an inverse trig function is an angle.

(b) Evaluate the following limit. Give your answer in radians. (5 pts.)

Answer: $\lim_{x \to -\infty} \tan^{-1}(e^{-x}) = \tan^{-1}(e^{\infty}) = \tan^{-1}(\infty) = \frac{\pi}{2}$ since the inverse tangent function has a horizontal asymptote at $y = \pi/2$ as $x \to \infty$.

(c) Find the linearization $L(x)$ of the function $f(x) = \sqrt[4]{1 - x}$ at the point $a = 0$. (5 pts.)
Answer: First write, $f(x) = (1 - x)^{1/4}$. Then, using the chain rule, we have $f'(x) = \frac{1}{4}(1 - x)^{-3/4} \cdot (-1)$. Thus, we compute that $f'(0) = -1/4$. Since $f(0) = 1$, the tangent line goes through the point $(0, 1)$. Therefore, the linearization at $a = 0$, which is just the equation of the tangent line to f through $(0, 1)$, is $L(x) = -\frac{1}{4}x + 1$.

(d) If $x(t) = 2t^3 - 21t^2 + 60t$ represents the position (in inches) of a particle traveling along the x-axis at time t (in seconds), find the total distance traveled by the particle in the first four seconds. (8 pts.)
Answer: 72 inches. First we compute the velocity $x'(t)$ to see if the particle changes directions in the first four seconds. We see that

$$x'(t) = 6t^2 - 42t + 60 = 6(t^2 - 7t + 10) = 6(t - 5)(t - 2),$$

which means the particle changes directions at time $t = 2$. It also changes directions at $t = 5$, but since this time is not in the interval $0 \leq t \leq 4$, we disregard it. The total distance traveled will be the distance traveled (to the right) over the first two seconds plus the distance traveled (to the left) over the next two seconds. This is given by

$$|x(2) - x(0)| + |x(4) - x(2)| = |52 - 0| + |32 - 52| = 52 + 20 = 72.$$