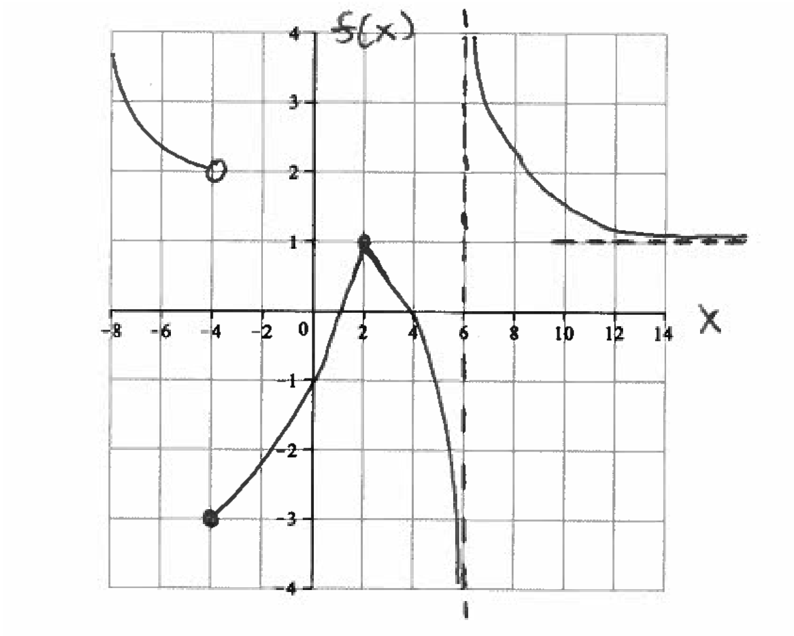
MATH 135 Calculus I

Exam #2 Solutions **October 30, 2013** **Prof. G. Roberts**

1. The graph of $f(x)$ is shown below. Use it to answer each of the following questions. Note that ∞ or $-\infty$ are acceptable answers for the limit problems. (17 pts.)



(a) Evaluate $\lim_{x \to -4^+} f(x)$

Answer: -3

(b) Evaluate $\lim_{x \to -4^-} f(x)$

Answer: 2

(c) Evaluate $\lim_{x \to 6^+} f(x)$

Answer: ∞

(d) Evaluate $\lim_{x \to \infty} f(x)$

Answer: 1 (horizontal asymptote at $y = 1$)

(e) List all the x-values where f is **not** continuous.

Answer: $x = -4$ (limit does not exist) and $x = 6$ (no function value)

(f) List all the x-values where f is **not** differentiable.

Answer: $x = 2$ (cusp) and $x = -4, 6$ (if f is not continuous at a point, then it cannot be differentiable there either)

2. Evaluate each of the following limits, if they exist. Note that ∞ or $-\infty$ are acceptable answers. Be sure to show your work. (5 pts. each)

(a) $\lim_{x \to \pi} \sqrt{2\cos^2(x) + 3}$

Answer: $\sqrt{5}$. Using the fact that the function is continuous at $x = \pi$, we simply plug in $x = \pi$ to obtain $\sqrt{2\cos^2(\pi) + 3} = \sqrt{2(-1)^2 + 3} = \sqrt{5}$.

(b) $\lim_{x \to 5} \dfrac{x^2 - 3x - 10}{x^2 - 25}$

Answer: $7/10$. Factor, simplify and then take the limit. After factoring, the limit becomes

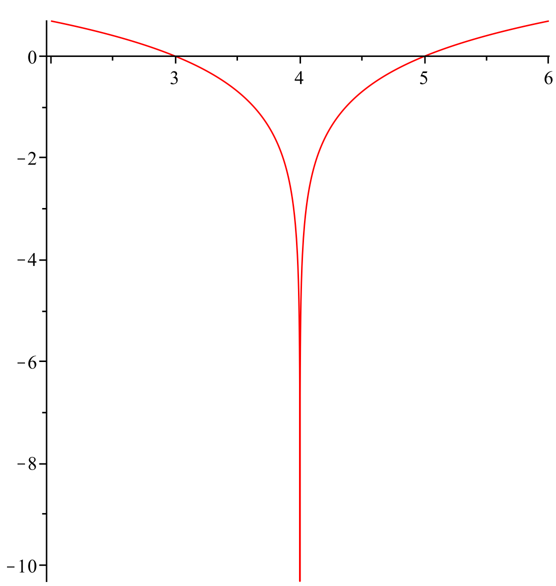
$$\lim_{x \to 5} \frac{(x-5)(x+2)}{(x-5)(x+5)} = \lim_{x \to 5} \frac{x+2}{x+5} = \frac{7}{10}.$$

(c) $\lim_{x \to -\infty} \dfrac{\pi - 2x^2 + 15x^4}{1 + 7x - 3x^4}$

Answer: -5. Since the highest powers in the numerator and denominator are equal, we simply take the ratio of their coefficients, $15/(-3) = -5$. The argument is made more rigorous by dividing top and bottom by the highest power x^4, and then using the limit laws to arrive at -5.

(d) $\lim_{x \to 4} \ln(|x - 4|)$

Answer: $-\infty$. One of the hardest problems on the exam. One approach is to draw the graph of the function. First, if $x > 4$, then $|x - 4| = x - 4$, so we want to sketch $\ln(x - 4)$ over the domain $x > 4$. This is the graph of $\ln x$ shifted right by 4 units. Then, for $x < 4$, we have $|x - 4| = -(x - 4)$, so we want to sketch the graph of $\ln(-(x - 4))$ over the domain $x < 4$. This is the graph of $\ln x$ reflected about the y-axis (replace x by $-x$) and then shifted right by 4 units (replace x by $x - 4$). The graph of $\ln(|x - 4|)$ is shown below, from which it is clear that the limit is $-\infty$. Since $\ln(x)$ has a vertical asymptote at $x = 0$, $\ln(|x - 4|)$ has a vertical asymptote at $x = 4$.



3. **(a)** State a limit definition for the derivative of a function $f(x)$ at the point $x = a$. (3 pts.)

$$f'(a) \;=\; \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \;=\; \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

(b) Use your limit definition from part **(a)** to find $f'(2)$ where $f(x) = \dfrac{3}{x^2}$. (9 pts.)

Answer: $f'(2) = -3/4$.

Method 1: Using the $\lim\limits_{h\to 0}$ definition, we have

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{\frac{3}{(2+h)^2} - \frac{3}{4}}{h} \\
&= \lim_{h\to 0} \frac{\frac{12-3(2+h)^2}{4(2+h)^2}}{\frac{h}{1}} \\
&= \lim_{h\to 0} \frac{12 - 3(4 + 4h + h^2)}{4h(2+h)^2} \\
&= \lim_{h\to 0} \frac{-12h - 3h^2}{4h(2+h)^2} \\
&= \lim_{h\to 0} \frac{-3h(4+h)}{4h(2+h)^2} \\
&= \lim_{h\to 0} \frac{-3(4+h)}{4(2+h)^2} \\
&= \frac{-3\cdot 4}{4(2)^2} \;=\; \frac{-12}{16} \;=\; -\frac{3}{4}.
\end{aligned}$$

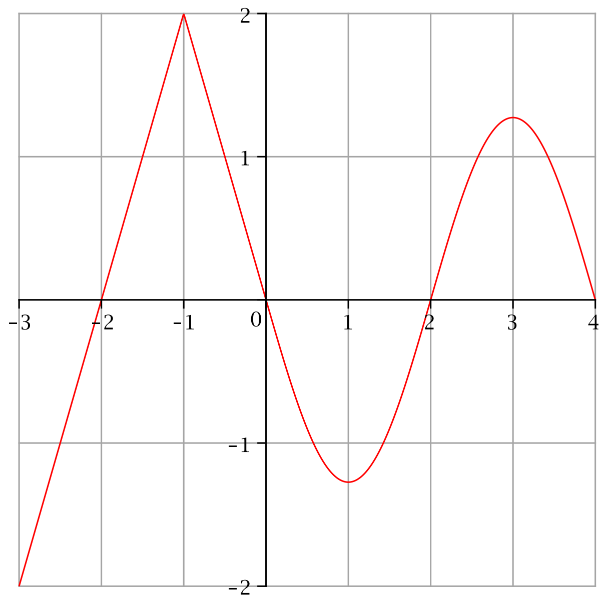
Method 2: Using the $\lim_{x\to a}$ definition, we have

$$\begin{aligned}
f'(2) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\\
&= \lim_{x\to 2}\frac{\frac{3}{x^2}-\frac{3}{4}}{x-2}\\
&= \lim_{x\to 2}\frac{\frac{12-3x^2}{4x^2}}{\frac{x-2}{1}}\\
&= \lim_{x\to 2}\frac{12-3x^2}{4x^2(x-2)}\\
&= \lim_{x\to 2}\frac{3(4-x^2)}{4x^2(x-2)}\\
&= \lim_{x\to 2}\frac{3(2-x)(2+x)}{4x^2(x-2)}\\
&= \lim_{x\to 2}\frac{-3(2+x)}{4x^2}\\
&= \frac{-3\cdot 4}{4(2)^2} = \frac{-12}{16} = -\frac{3}{4}.
\end{aligned}$$

(c) Find the equation of the tangent line to $f(x)=\dfrac{3}{x^2}$ at the point $x=2$. (5 pts.)

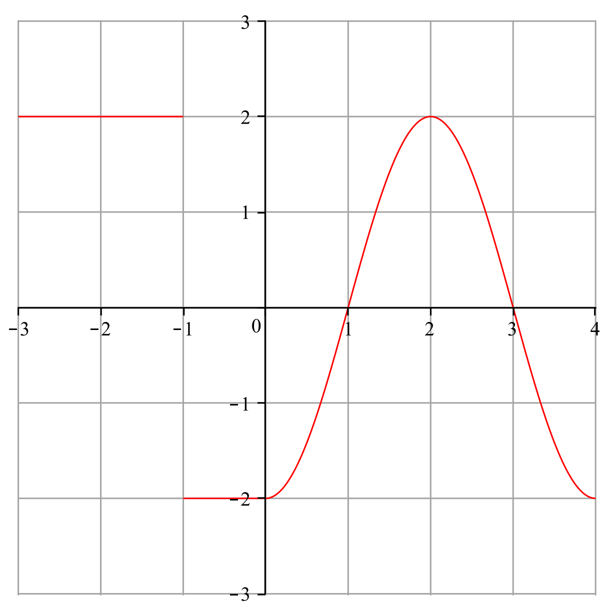
Answer: $y=-\frac{3}{4}x+\frac{9}{4}$. From part **(b)**, we have that $m=f'(2)=-3/4$. Thus, $y=-\frac{3}{4}x+b$. To find b, we use the point $(2,3/4)$ since $x=2$ implies $y=f(2)=3/4$. Therefore, we have $3/4=-\frac{3}{4}\cdot 2+b$ or $3/4=-3/2+b$, which implies $b=9/4$. Thus the equation of the tangent line is $y=-\frac{3}{4}x+\frac{9}{4}$.

4. Given the graph of the function $g(x)$ below, sketch the graph of the derivative $g'(x)$ on the axes provided. (12 pts.)



Answer:

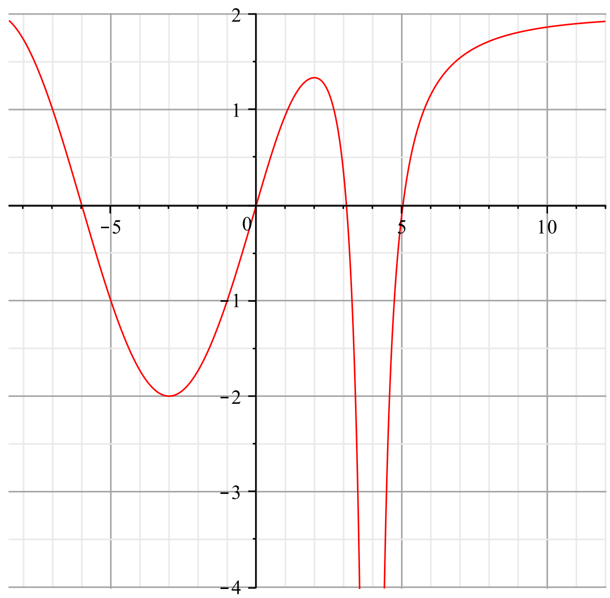
Note that the graph of the derivative will have a hole at $x = -1$ as $g'(-1)$ does not exist (corner).



5. Sketch the graph of a function $f(x)$ satisfying all of the following properties: (12 pts.)

- f is continuous at all x except for $x = 4$
- f has a vertical asymptote at $x = 4$
- $f(0) = 0$
- $f'(-3) = 0$ and $f'(2) = 0$
- $\lim_{x\to\infty} f(x) = 2$
- $f'(x) < 0$ if $x < -3$ or $2 < x < 4$
- $f'(x) > 0$ if $-3 < x < 2$ or $x > 4$
- $f''(x) < 0$ if $x < -6$ or $0 < x < 4$ or $x > 4$
- $f''(x) > 0$ if $-6 < x < 0$

Answer:



6. Some final conceptual questions. You must show your work to receive any partial credit. (22 pts.)

(a) If $6x - 4 \le h(x) \le x^2 + 5$ for all x, find $\lim_{x\to 3} h(x)$.

Answer: Using the Squeeze Theorem, since $\lim_{x\to 3} 6x - 4 = 14$ and $\lim_{x\to 3} x^2 + 5 = 14$, we have that $\lim_{x\to 3} h(x) = 14$.

(b) Suppose that $P(s)$ represents the profit earned in dollars for selling s stereos. Which of the following best describes the meaning of $P'(500) = 100$?

(i) The profit earned from selling 100 stereos is \$500.
(ii) The profit earned from selling 500 stereos is \$100.
(iii) Selling the 501st stereo will earn, approximately, an additional \$100 in profit.
(iv) Selling the 101st stereo will earn, approximately, an additional \$500 in profit.
(v) The rate of change of the profit is \$500 per stereo after selling 100 stereos.

Answer: (iii)

(c) When trying to assuage the fears of the American people concerning inflation, former U.S. President Nixon once stated, "Although the rate of inflation is increasing, it is increasing at a decreasing rate." Interpret this statement by determining the signs (positive, negative or zero) of $r'(t)$ and $r''(t)$, where $r(t)$ represents the rate of inflation at time t.

Answer: The first derivative $r'(t)$ is positive because the rate of inflation is increasing. However, since it is increasing at a decreasing rate (concave down), the second derivative $r''(t)$ is negative. As a point of interest, since $r(t)$ is really a "rate," it is actually a derivative in its own right. Thus, Nixon was actually speaking to the public about the third derivative!

(d) Find and simplify $f''(x)$ if $f(x) = \sqrt{x}\, e^x$.

Answer: We write $f(x) = x^{1/2} \cdot e^x$. By the product rule, we have

$$f'(x) \;=\; \frac{1}{2}x^{-1/2} \cdot e^x + x^{1/2} \cdot e^x \;=\; e^x(\frac{1}{2}x^{-1/2} + x^{1/2}).$$

Using the product rule again, we find that

$$\begin{aligned} f''(x) &= e^x(\frac{1}{2}x^{-1/2} + x^{1/2}) + e^x(-\frac{1}{4}x^{-3/2} + \frac{1}{2}x^{-1/2}) \\ &= e^x\left(\frac{1}{2}x^{-1/2} + x^{1/2} - \frac{1}{4}x^{-3/2} + \frac{1}{2}x^{-1/2}\right) \\ &= e^x\left(x^{-1/2} + x^{1/2} - \frac{1}{4}x^{-3/2}\right) \quad \text{or} \quad e^x\left(\sqrt{x} + \frac{1}{\sqrt{x}} - \frac{1}{4x\sqrt{x}}\right). \end{aligned}$$

(e) Suppose that $Q(x) = \dfrac{e^x}{g(x)}$ and that $g(1) = 3, g'(1) = -5$. Find $Q'(1)$ (give the **exact** answer, no decimals).

Answer: $8e/9$. By the quotient rule, we have

$$Q'(x) \;=\; \frac{g(x) \cdot e^x - e^x \cdot g'(x)}{(g(x))^2}.$$

Now plug in $x = 1$ and simplify. This yields

$$\begin{aligned} Q'(1) &= \frac{g(1) \cdot e^1 - e^1 \cdot g'(1)}{(g(1))^2} \\ &= \frac{3e - e(-5)}{9} = \frac{8e}{9}. \end{aligned}$$