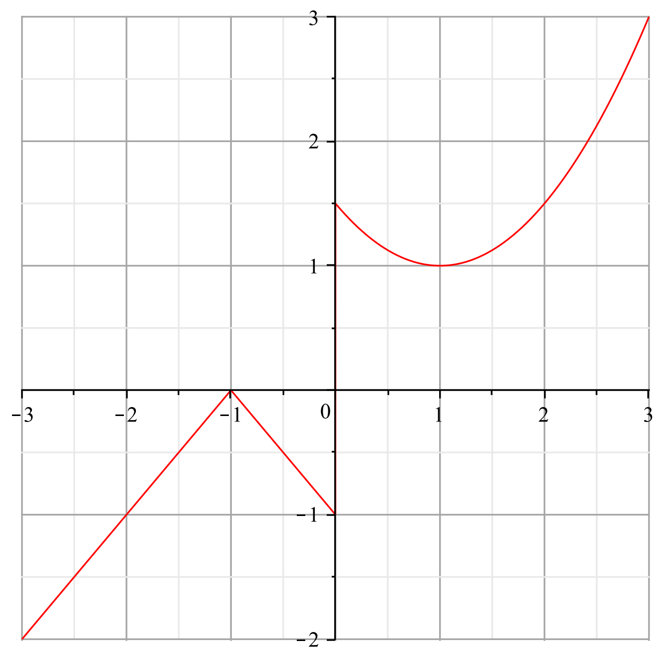
# MATH 135 Calculus 1

**Exam #1 SOLUTIONS** **September 25, 2013** **Prof. G. Roberts**

1. The **ENTIRE** graph of $f(x)$ is shown below. Use it to answer each of the following questions: (20 pts.)




**(a)** What is the domain of $f$? $[-3, 3]$

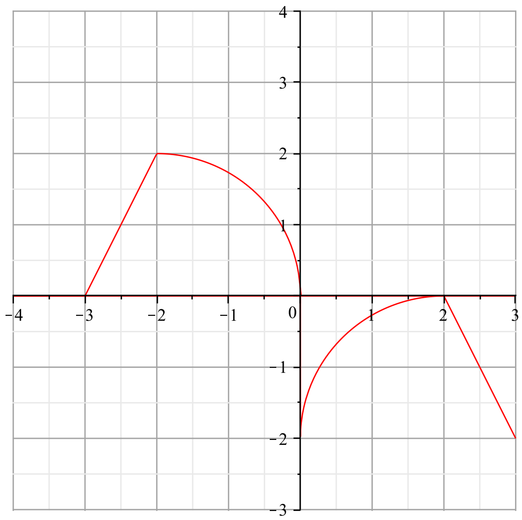
**(b)** What is the range of $f$? $[-2, 0] \cup [1, 3]$ Note that there is a gap in the function and that no $y$-values between 0 and 1 have pre-images in the domain.

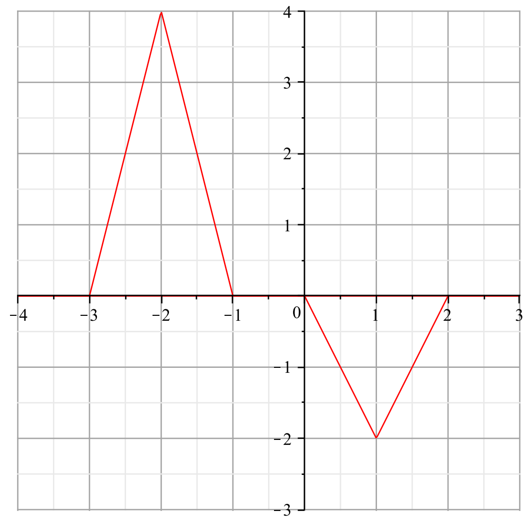
**(c)** Is $f$ a one-to-one function? Explain. No, it fails the horizontal line test. Specifically, there are output values $y$ of the function that have more than one pre-image $x$ in the domain. For example, $f(0) = f(2) = 1.5$.

**(d)** For what value(s) of $x$ does $f(x) = 1.5$? $x = 0$ and $x = 2$

**(e)** What is $(f \circ f)(-2)$? 0
$(f \circ f)(-2) = f(f(-2)) = f(-1) = 0.$

2. For each of the graphs shown below, one or more transformations have been applied to the original function $f(x)$ to obtain a new function $g(x)$. In mathematical terms, state the formula for $g(x)$ in terms of $f(x)$. For example, a typical answer might be $g(x) = 3f(2x+1)$. (12 pts.)




**(a)** $g(x) = \underline{f(-x) - 2}$ **(b)** $g(x) = \underline{-\frac{1}{2}f(x-3)}$

For part **(a)**, we reflect the graph about the $y$-axis ($f(-x)$) and shift it down by two units, giving $f(-x) - 2$.

For part **(b)**, we shift the graph of $f(x)$ right three units ($f(x-3)$) and compress it by a factor of 2, giving $\frac{1}{2}f(x-3)$. We then reflect it about the $x$-axis, using $-\frac{1}{2}f(x-3)$.

3. A population of mold has begun to develop on a piece of bread you brought back to your dormitory from Kimball dining hall. Initially, the population is only 150 cells, but after five hours, it has increased to 600 cells. In the questions below use the variable $t$ for time (measured in hours) and $P$ for the population of mold cells.

   **(a)** Using a **linear** model, find a linear function that models the population $P$ of mold cells on the bread as a function of time $t$. (6 pts.)

   **Answer:** Using the variables $(t, P)$, the two data points given in the problem are $(0, 150)$ and $(5, 600)$. Thus, the slope is found by $(600 - 150)/(5 - 0) = 450/5 = 90$. The $P$-intercept is simply 150. Therefore, a linear equation for the population of mold cells is

$$P(t) \;=\; 90t + 150.$$

   **(b)** Using an **exponential** growth model, find an exponential function that models the population $P$ of mold cells on the bread as a function of time $t$. (6 pts.)

   **Answer:** Recall that an exponential model is of the form $P(t) = P_0 \cdot a^t$. We have that $P_0 = 150$. Then, using $t = 5, P = 600$ as a second data point, we have $600 = 150 \cdot a^5$. This reduces to $a^5 = 4$ or $a = 4^{1/5}$. Thus, the formula for the number of cells after $t$ hours is

$$P(t) \;=\; 150(4^{1/5})^t = 150 \cdot 4^{t/5}.$$

   A simpler way to get right to this formula is to notice that the population quadruples in 5 hours, so that $a = 4$ and $t$ is replaced by $t/5$.

**(c)** Public Safety will confiscate your bread if the number of mold cells reaches $150,000$. Using your exponential model from part **(b)**, after how many hours will this happen? Give the **exact** answer as well as an approximation rounded to two decimal places. (6 pts.)

**Answer:** Using the expression derived in part **(b)**, we must solve the equation $150,000 = 150 \cdot 4^{t/5}$ for $t$. The first step is to divide both sides by 150. This gives $1000 = 4^{t/5}$. To solve this we take the natural logarithm of both sides, $\ln(1000) = \ln(4^{t/5}) = (t/5) \cdot \ln(4)$. Dividing both sides by $\ln 4$ and multiplying by 5 gives

$$t = \frac{5\ln(1000)}{\ln(4)},$$

which is the **exact** answer. Using a calculator and rounding to the nearest hundredth, we find that $t \approx 24.91$ hours.

4. Consider the following parametric equations:

$$x = 4 - 2t, \quad y = 4t^2 - 3, \quad -2 \leq t \leq 2$$

**(a)** At what point in the $xy$-plane does the curve traced out by these equations begin? At what point does the curve end? (4 pts.)

**Answer:** The curve starts at the point $(8, 13)$, which is found by plugging in $t = -2$ into the $x$ and $y$ equations, respectively. In other words, $x = 4 - 2(-2) = 8$ and $y = 4(-2)^2 - 3 = 13$. The curve ends at the point $(0, 13)$, which is found by plugging in $t = 2$ into the $x$ and $y$ equations, respectively.

**(b)** Eliminate the parameter to find an equation for the curve using only $x$ and $y$ variables. Be sure to simplify your answer. (6 pts.)

**Answer:** We first solve the $x$ equation for $t$. We have $x = 4 - 2t$ or $x - 4 = -2t$, which implies $t = (x-4)/(-2)$. Next, we substitute this expression into the $y$ equation yielding

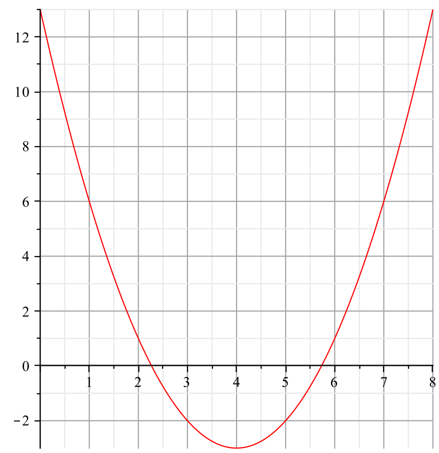
$$\begin{aligned} y &= 4\left(\frac{x-4}{-2}\right)^2 - 3 \\ &= 4 \cdot \frac{(x-4)^2}{4} - 3 \\ &= (x-4)^2 - 3 \quad \text{or} \quad x^2 - 8x + 13 \end{aligned}$$

**(c)** What type of curve is being traced out? (4 pts.)

**Answer:** The curve being traced out is a parabola.

**(d)** Sketch the curve being traced out in the $xy$-plane. Be sure to indicate with an arrow which direction the curve is being traced out as the parameter increases. (6 pts.)

**Answer:** The curve is a parabola that moves from right to left. It starts at $(8, 13)$ at time $t = -2$, passes through the points $(6, 1)$ at $t = -1$, $(4, -3)$ (vertex) at $t = 0$, and $(2, 1)$ at $t = 1$, before ending at the point $(0, 13)$ at time $t = 2$ (see graph below).




5. Some final conceptual questions. You must show your work to receive any partial credit. (30 pts.)

   **(a)** If $s(t)$ represents the position function of a moving object, then the instantaneous velocity at the time $t = a$ is defined as the slope of the tangent to the graph of $s(t)$ at the point $t = a$.

   **(b)** The period of the function $g(x) = 4\sin(3\pi x)$ is 2/3.
   **Answer:** Using the formula that period is $2\pi/b$, we substitute $b = 3\pi$ and simplify to get 2/3.

   **(c)** If $h(x)$ is an invertible function such that $h(4) = -7$, then $h^{-1}(-7) = 4$ .
   **Answer:** This is just the definition of an inverse function.

   **(d)** The function $g(x) = \cos x + x^2 - 3$ is even.
   (odd, even, neither odd nor even, both odd and even)
   **Answer:** Since $\cos x, x^2$ and $-3$ are all even functions, the sum is also even. In other words, $g(-x) = g(x)$ and thus $g$ is an even function.

   **(e)** Find the domain and range of the function $g(x) = \ln(x - 3)$.
   **Answer:** The domain is $x > 3$ or the interval $(3, \infty)$. The range is all real numbers or $\mathbb{R}$. These answers are obtained easily by shifting the graph of $\ln x$ to the right 3 units.

   **(f)** Find the **exact** solution to the equation $\ln(5 + 2x) = \pi$.
   **Answer:** The **exact** solution is $x = (e^\pi - 5)/2$. The first step is to raise both sides to the base $e$. This gives

   $$e^{\ln(5+2x)} = e^\pi \quad \text{or} \quad 5 + 2x = e^\pi.$$

   Next, subtract 5 from both sides and then divide by 2. There is no need (or use) for a calculator on this problem.

   **(g)** If $s(t) = 3t^2 - 5t$ represents the distance in feet a ball has traveled after $t$ seconds, then the average velocity over the interval $[1, 4]$ is ____________________.
   **Answer:** 10 feet per second.
   Using the formula average velocity is $(s(t_2) - s(t_1))/(t_2 - t_1)$, we compute the average velocity to be

   $$\frac{s(4) - s(1)}{4 - 1} = \frac{28 - (-2)}{3} = \frac{30}{3} = 10.$$