# MATH 134 Calculus 2 with FUNdamentals <br> Sample Final Exam Questions 

## Solutions

1. Define $A(x)=\int_{0}^{x} f(t) d t$ for $0 \leq x \leq 5$, where the graph of $f(t)$ is given below.

(a) Find $A(0), A(3)$ and $A(5)$.

Answer: $A(0)=0, A(3)=5, A(5)=5$. The function $A(x)$ represents the area under the curve from 0 to $x$. Thus $A(0)=0$ and $A(3)=\frac{1}{2} \cdot 1 \cdot 2+2 \cdot 2=1+4=5$ (area of triangle plus area of square). $A(5)=5$ as well because the area over $[3,4]$ is equal to the area over $[4,5]$ but with an opposite sign (so they cancel out).
(b) Find $A^{\prime}(1)$ if it exists. If it does not exist, explain why.

Answer: $A^{\prime}(1)=2$. Using FTC, part 2, we have that $A^{\prime}(x)=f(x)$ and thus $A^{\prime}(1)=$ $f(1)=2$, as can be seen from the graph.
(c) Find $A^{\prime \prime}(1)$ if it exists. If it does not exist, explain why.

Answer: $A^{\prime \prime}(1)$ does not exist. Since $A^{\prime}(x)=f(x)$, we have that $A^{\prime \prime}(x)=f^{\prime}(x)$ (differentiate both sides with respect to $x$ ). This means that $A^{\prime \prime}(1)=f^{\prime}(1)$ does not exist because there is a corner in the graph of $f$ at $t=1$.
(d) Find the intervals on which $A(x)$ is increasing and decreasing.

Answer: Increasing on $(0,4)$; decreasing on $(4,5)$. Since $A^{\prime}(x)=f(x)$ and since a function is increasing whenever its derivative is positive, we see that $A$ is increasing whenever $f>0$ (graph above the axis), or when $0<x<4 . A(x)$ is decreasing whenever $f(x)<0$ (graph below the axis) or $4<x<5$.
(e) Find the intervals on which $A(x)$ is concave up and concave down.

Answer: Concave up on $(0,1)$; concave down on $(3,5)$. Since $A^{\prime \prime}(x)=f^{\prime}(x)$ and since a function is concave up whenever its second derivative is positive, we see that $A$ is concave up whenever $f^{\prime}>0$ (positive slope), or when $0<x<1$, while $A$ is concave down whenever $f^{\prime}<0$ (negative slope), or when $3<x<5$. Note that $A^{\prime \prime}(x)=f^{\prime}(x)=0$ when $1<x<3$ because the graph is a horizontal line segment.
2. Evaluate the following integrals:
(a) $\int \sqrt{2 x+1}+\sin (3 x) d x$

Answer: The first function can be integrated as a $u$-sub with $u=2 x+1$ and $d u=2 d x$ or $d x=\frac{1}{2} d u$. We have

$$
\int \sqrt{2 x+1} d x=\int \sqrt{u} \cdot \frac{1}{2} d u=\frac{1}{2} \int u^{1 / 2} d u=\frac{1}{2} \cdot \frac{2}{3} u^{3 / 2}+c=\frac{1}{3}(2 x+1)^{3 / 2}+c
$$

Then, using the rule for integrating $\sin (k x)$, we have $\int \sin (3 x) d x=-\frac{1}{3} \cos (3 x)$. By linearity, the integral is the sum of these two antiderivatives:

$$
\frac{1}{3}(2 x+1)^{3 / 2}-\frac{1}{3} \cos (3 x)+c
$$

(b) $\int \sec ^{2}(2 \theta) e^{\tan (2 \theta)} d \theta$

Answer: This integral can be done using the $u$-substitution $u=\tan (2 \theta)$. Then $d u=$ $2 \sec ^{2}(2 \theta) d \theta$ (chain rule) or $\sec ^{2}(2 \theta) d \theta=\frac{1}{2} d u$. We compute

$$
\begin{aligned}
\int \sec ^{2}(2 \theta) e^{\tan (2 \theta)} d \theta & =\int e^{u} \cdot \frac{1}{2} d u \\
& =\frac{1}{2} \int e^{u} d u \\
& =\frac{1}{2} e^{u}+c \\
& =\frac{1}{2} e^{\tan (2 \theta)}+c .
\end{aligned}
$$

(c) $\int t^{6} \ln t d t$

Answer: Use integration by parts. Let $u=\ln t$ and $d v=t^{6} d t$. This leads to $d u=\frac{1}{t} d t$ and $v=\frac{1}{7} t^{7}$. The integration by parts formula then yields

$$
\begin{aligned}
\int t^{6} \ln t d t & =\frac{1}{7} t^{7} \ln t-\int \frac{1}{7} t^{7} \cdot \frac{1}{t} d t \\
& =\frac{1}{7} t^{7} \ln t-\frac{1}{7} \int t^{6} d t \\
& =\frac{1}{7} t^{7} \ln t-\frac{1}{49} t^{7}+c \\
& =\frac{t^{7}}{49}(7 \ln t-1)+c
\end{aligned}
$$

(d) $\int \frac{z^{2}}{\sqrt{1-z^{2}}} d z$

Answer: This integral can be computed with the trig substitution $z=\sin \theta$. Then we have $d z=\cos \theta d \theta$ and $z^{2}=\sin ^{2} \theta$. Using the identity $\cos ^{2} \theta+\sin ^{2} \theta=1$, we find

$$
\sqrt{1-z^{2}}=\sqrt{1-\sin ^{2} \theta}=\sqrt{\cos ^{2} \theta}=\cos \theta
$$

Thus, the integral transforms to

$$
\begin{aligned}
\int \frac{\sin ^{2} \theta}{\cos \theta} \cdot \cos \theta d \theta & =\int \sin ^{2} \theta d \theta \\
& =\int \frac{1}{2}(1-\cos (2 \theta)) d \theta \quad \text { (identity from Section 7.2) } \\
& =\frac{1}{2}\left(\theta-\frac{1}{2} \sin (2 \theta)\right)+c \\
& \left.=\frac{1}{2}(\theta-\sin \theta \cos \theta)+c \quad \text { (trig identity } \sin (2 \theta)=2 \sin \theta \cos \theta\right)
\end{aligned}
$$

To finish the problem, we return to the original variable $z$. We have $\sin \theta=z$, so $\theta=\sin ^{-1} z$. The final step comes from using SOH-CAH-TOA and the Pythagorean Theorem, beginning with $\sin \theta=\frac{z}{1}$ (see figure below). Hence, the solution is

$$
\frac{1}{2}\left(\sin ^{-1} z-z \sqrt{1-z^{2}}\right)+c
$$


(e) $\int \frac{21}{2 x^{2}+5 x-3} d x$

Answer: The denominator factors as $(2 x-1)(x+3)$ so this suggests partial fractions as an appropriate technique. We seek constants $A$ and $B$ such that

$$
\frac{21}{(2 x-1)(x+3)}=\frac{A}{2 x-1}+\frac{B}{x+3} .
$$

Multiplying through by the LCD $(2 x-1)(x+3)$ gives

$$
21=A(x+3)+B(2 x-1) .
$$

Next we plug in the roots $x=1 / 2$ and $x=-3$. Using $x=1 / 2$ in the previous equation, we find $21=7 A / 2$ or $A=6$. Likewise, setting $x=-3$ in the previous equation gives
$21=-7 B$ or $B=-3$. Thus, the integral transforms into

$$
\begin{aligned}
\int \frac{6}{2 x-1}+\frac{-3}{x+3} d x & =6 \cdot \frac{1}{2} \ln |2 x-1|-3 \ln |x+3|+c \quad(u=2 x-1, d u=2 d x) \\
& =3 \ln |2 x-1|-3 \ln |x+3|+c \\
& =3 \ln \left|\frac{2 x-1}{x+3}\right|+c
\end{aligned}
$$

3. Approximate the value of the integral $\int_{1}^{3} \cos \left(x^{2}\right) d x$ using the given rule:
(a) Left-hand Sum $L_{4}$

Answer: $L_{4}=0.128967$. Let us define $g(x)=\cos \left(x^{2}\right)$. The width of each rectangle is $\Delta x=(3-1) / 4=1 / 2$. Draw a number line and divide the interval $[1,3]$ into four equal subintervals, each of width $1 / 2$. This gives the subintervals [1, 1.5], [1.5, 2], $[2,2.5],[2.5,3]$. Evaluating $g$ at the left endpoints of each subinterval gives an estimated area of

$$
\begin{aligned}
L_{4} & =\frac{1}{2}(g(1)+g(1.5)+g(2)+g(2.5)) \\
& =\frac{1}{2}\left(\cos (1)+\cos \left(1.5^{2}\right)+\cos (4)+\cos \left(2.5^{2}\right)\right) \\
& \approx 0.128967
\end{aligned}
$$

Be sure to set your calculator to radians!
(b) Midpoint Rule $M_{4}$

Answer: $M_{4}=-0.179102$. The width of each rectangle is still $1 / 2$. Evaluating $g$ at the midpoints of each subinterval gives an estimated area of

$$
\begin{aligned}
M_{4} & =\frac{1}{2}(g(1.25)+g(1.75)+g(2.25)+g(2.75)) \\
& =\frac{1}{2}\left(\cos \left(1.25^{2}\right)+\cos \left(1.75^{2}\right)+\cos \left(2.25^{2}\right)+\cos \left(2.75^{2}\right)\right) \\
& \approx-0.179102 .
\end{aligned}
$$

4. Let $R$ be the region in the first quadrant bounded by $y=\sqrt{x}$ and $y=x^{2}$.
(a) Sketch the region $R$ and find its area.

## Answer:

The curves are the top half of a parabola opening to the right and the standard parabola opening upwards. To find where the two curves intersect, we solve $\sqrt{x}=x^{2}$ or $x=x^{4}$. This is equivalent to $x^{4}-x=0$ or $x\left(x^{3}-1\right)=0$ which gives $x=0$ or $x=1$. The curves intersect at $(0,0)$ and $(1,1)$ (see Figure 1).


Figure 1: The region $R$.

We find the area of $R$ by integrating the difference of the top function and the bottom one from $x=0$ to $x=1$. We compute

$$
\begin{aligned}
A & =\int_{0}^{1} \sqrt{x}-x^{2} d x \\
& =\int_{0}^{1} x^{1 / 2}-x^{2} d x \\
& =\frac{2}{3} x^{3 / 2}-\left.\frac{x^{3}}{3}\right|_{0} ^{1} \\
& =\frac{2}{3}-\frac{1}{3}-0 \\
& =\frac{1}{3}
\end{aligned}
$$

(b) Find the volume of the solid of revolution obtained by rotating $R$ about the $x$-axis.


Answer: $3 \pi / 10$

We use the washer method since there is a gap between $R$ and the axis of rotation. The outer radius is $\sqrt{x}$ (distance between the right-facing parabola and the $x$-axis) and the inner radius is $x^{2}$ (distance between the upward-facing parabola and the $x$-axis). See the above figure, generated using desmos.com. Thus, the volume is given by

$$
\begin{aligned}
\pi \int_{0}^{1}(\sqrt{x})^{2}-\left(x^{2}\right)^{2} d x & =\pi \int_{0}^{1} x-x^{4} d x \\
& =\pi\left(\frac{x^{2}}{2}-\left.\frac{x^{5}}{5}\right|_{0} ^{1}\right) \\
& =\pi\left(\frac{1}{2}-\frac{1}{5}-0\right)=\frac{3 \pi}{10}
\end{aligned}
$$

(c) Find the volume of the solid of revolution obtained by rotating $R$ about the line $y=1$.

Answer: $11 \pi / 30$
We use the washer method since there is a gap between $R$ and the axis of rotation. The outer radius in this case is $1-x^{2}$ (distance between the line $y=1$ and the the upwardfacing parabola) and the inner radius is $1-\sqrt{x}$ (distance between the line $y=1$ and the right-facing parabola). Thus, the volume is given by

$$
\begin{aligned}
\pi \int_{0}^{1}\left(1-x^{2}\right)^{2}-(1-\sqrt{x})^{2} d x & =\pi \int_{0}^{1} 1-2 x^{2}+x^{4}-(1-2 \sqrt{x}+x) d x \\
& =\pi \int_{0}^{1} x^{4}-2 x^{2}-x+2 \sqrt{x} \\
& =\pi\left(\frac{x^{5}}{5}-\frac{2 x^{3}}{3}-\frac{x^{2}}{2}+\left.\frac{4}{3} x^{3 / 2}\right|_{0} ^{1}\right) \\
& =\pi\left(\frac{1}{5}-\frac{2}{3}-\frac{1}{2}+\frac{4}{3}-0\right)=\frac{11 \pi}{30}
\end{aligned}
$$

5. Sequences and Series:
(a) Find a formula for the general term $a_{n}$ (start with $n=1$ ) for the sequence

$$
-\frac{1}{2}, \frac{1}{4},-\frac{1}{8}, \frac{1}{16},-\frac{1}{32},-+\ldots
$$

Answer: $a_{n}=\left(-\frac{1}{2}\right)^{n}$. Notice that the signs alternate and that the denominators are successive powers of 2 .
(b) Does the sequence given by $a_{n}=\ln \left(\frac{1+e n^{3}}{4+n^{3}}\right)$ converge or diverge? If it converges, find the limit.

Answer: The sequence converges to 1. This follows by first evaluating the limit inside
the $\ln$ function. We have $\lim _{n \rightarrow \infty} \frac{1+e n^{3}}{4+n^{3}}=e$. Since the highest power in the numerator and denominator are the same, we just take the ratio of their coefficients (one could also use L'Hôpital's Rule three times). It follows that the sequence is approaching $\ln (e)=1$ as $n \rightarrow \infty$.
(c) Find the sum of the geometric series: $18-6+2-2 / 3+2 / 9-+\cdots$

Answer: This is a geometric series with ratio $r=-1 / 3$ and starting term 18. The ratio can be found by inspection or by dividing any term by the term proceeding it (e.g., $-6 \div 18=-1 / 3$, or $-\frac{2}{3} \div 2=-1 / 3$ ). Then, using the formula for the sum of a geometric series, we have

$$
S=\frac{a}{1-r}=\frac{18}{1-(-1 / 3)}=\frac{18}{\frac{4}{3}}=\frac{27}{2}
$$

6. Determine whether the given infinite series converges or diverges using any of the tests from class or the text. You must provide a valid reason to receive full credit.
(a) $\sum_{n=1}^{\infty} \frac{n^{5}}{100 n^{5}+1}$

Answer: This series diverges by the $n$th term test. We have

$$
\lim _{n \rightarrow \infty} \frac{n^{5}}{100 n^{5}+1}=\lim _{n \rightarrow \infty} \frac{n^{5}}{100 n^{5}}=\frac{1}{100} \neq 0
$$

and thus the series diverges by the $n$th term test.
(b) $\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$

Answer: This series converges by the $p$-series test. Since $p=1.01>1$, the series converges by the $p$-series test.
(c) $\sum_{n=1}^{\infty} n e^{-2 n}$

Answer: This series converges by the integral test. Let $f(x)=x e^{-2 x}$. This function is positive, decreasing, and continuous for $x \geq 1$. It is decreasing because $f^{\prime}(x)=$ $e^{-2 x}+x \cdot-2 e^{-2 x}=e^{-2 x}(1-2 x)$ is negative for $x>1 / 2$.
To compute the integral, use integration by parts with $u=x$ and $d v=e^{-2 x} d x$. Then $d u=1 d x$ and $v=-\frac{1}{2} e^{-2 x}$. Recall that $\int e^{k x} d x=\frac{1}{k} e^{k x}+c$ and $\lim _{b \rightarrow \infty} e^{-k b}=0$ whenever $k>0$ (exponential decay; the graph of $e^{-k b}$ has a horizontal asymptote at $y=0$ ).

We have

$$
\begin{aligned}
\int_{1}^{\infty} x e^{-2 x} d x & =\lim _{b \rightarrow \infty} \int_{1}^{b} x e^{-2 x} d x \\
& =\lim _{b \rightarrow \infty}-\left.\frac{1}{2} x e^{-2 x}\right|_{1} ^{b}-\int_{1}^{b}-\frac{1}{2} e^{-2 x} d x \\
& =\lim _{b \rightarrow \infty}-\frac{1}{2} b e^{-2 b}+\frac{1}{2} e^{-2}-\left.\frac{1}{4} e^{-2 x}\right|_{1} ^{b} \\
& =\lim _{b \rightarrow \infty}-\frac{1}{2} b e^{-2 b}+\frac{1}{2} e^{-2}-\frac{1}{4} e^{-2 b}+\frac{1}{4} e^{-2} \\
& =\frac{3}{4} e^{-2} .
\end{aligned}
$$

The first term in the limit can be computed using L'Hôpital's Rule:

$$
\begin{aligned}
\lim _{b \rightarrow \infty}-\frac{1}{2} b e^{-2 b} & =\lim _{b \rightarrow \infty}-\frac{b}{2 e^{2 b}} \\
& =\lim _{b \rightarrow \infty}-\frac{1}{4 e^{2 b}} \\
& =0
\end{aligned}
$$

Since the integral converges to $\frac{3}{4} e^{-2}$, the infinite series converges.
(d) $\sum_{n=1}^{\infty}(-1)^{n+1}$

Answer: The series diverges by either the $n$th term test or the geometric series test. Notice that the terms of the series oscillate between 1 (if $n$ is odd) and -1 (when $n$ is even). It follows that $\lim _{n \rightarrow \infty}(-1)^{n+1}$ does not exist. By the $n$th term test, the series diverges. Alternatively, the series

$$
\sum_{n=1}^{\infty}(-1)^{n+1}=1-1+1-1+1-1+-\cdots
$$

can be interpreted as an infinite geometric series with ratio $r=-1$. Since $|-1|=1 \geq 1$, the series diverges by the geometric series test.
7. Find the solution to the given initial-value problems:
(a) $\frac{d y}{d t}=\frac{y}{1+t^{2}}, \quad y(0)=3$.

Answer: First we separate the variables:

$$
\frac{d y}{d t}=\frac{y}{1+t^{2}} \quad \Longrightarrow \quad \frac{1}{y} d y=\frac{1}{1+t^{2}} d t
$$

Next we integrate both sides and solve for $y$ :

$$
\begin{aligned}
\int \frac{1}{y} d y=\int \frac{1}{1+t^{2}} d t & \Longrightarrow \ln |y|=\tan ^{-1} t+c \\
& \Longrightarrow|y|=e^{\tan ^{-1} t+c}=e^{\tan ^{-1} t} \cdot e^{c} \\
& \Longrightarrow y=c e^{\tan ^{-1} t}
\end{aligned}
$$

To find the particular solution satisfying $y(0)=3$, we plug in $t=0$ and $y=3$ into the general solution we just found and solve for the constant $c$. This gives

$$
3=c e^{\tan ^{-1}(0)} \quad \Longrightarrow \quad 3=c e^{0} \quad \Longrightarrow \quad c=3
$$

Therefore, the particular solution is $y=3 e^{\tan ^{-1} t}$.
(b) $\frac{d y}{d x}=-2 x^{2} e^{y}, \quad y(3)=-\ln 9$.

Answer: First we separate the variables:

$$
\frac{d y}{d t}=-2 x^{2} e^{y} \quad \Longrightarrow \frac{1}{e^{y}} d y=-2 x^{2} d x
$$

Next we integrate both sides and solve for $y$ :

$$
\begin{aligned}
\int e^{-y} d y=\int-2 x^{2} d x & \Longrightarrow-e^{-y}=-\frac{2}{3} x^{3}+c \\
& \Longrightarrow e^{-y}=\frac{2}{3} x^{3}+c \\
& \Longrightarrow-y=\ln \left(\frac{2}{3} x^{3}+c\right) \\
& \Longrightarrow y=-\ln \left(\frac{2}{3} x^{3}+c\right)
\end{aligned}
$$

Note that we cannot bring the $c$ outside the parentheses because $\ln (a+c) \neq \ln a+\ln c$.
To find the particular solution satisfying $y(3)=-\ln 9$, we plug in $x=3$ and $y=-\ln 9$ into the general solution we just found and solve for the constant $c$. This gives

$$
-\ln 9=-\ln \left(\frac{2}{3} \cdot 3^{3}+c\right) \quad \Longrightarrow \quad \ln 9=\ln (18+c) \quad \Longrightarrow \quad 9=18+c \quad \Longrightarrow \quad c=-9 .
$$

Therefore, the particular solution is $y=-\ln \left(\frac{2}{3} x^{3}-9\right)$.
8. Suppose that Auntie Pat is cooking her Thanksgiving turkey (tofurkey for you vegetarians) for friends and family. The guests are planning to arrive at 5:00 pm. She preheats the oven to $400^{\circ} \mathrm{F}$. Suppose the initial temperature of the turkey is $50^{\circ} \mathrm{F}$. She places the turkey in the oven at 10:00 am. By noon the turkey has cooked to a temperature of $80^{\circ} \mathrm{F}$. Using Newton's law of cooling (or warming), at what time (to the nearest minute) will the temperature of the turkey be $150^{\circ} \mathrm{F}$ (medium rare and ready to serve)? Assume that the oven has a constant temperature of $400^{\circ} \mathrm{F}$ throughout the cooking. Does she make it in time for the guests or will she be serving hors d'ouvres for a while?

Answer: Let $y(t)$ be the temperature of the turkey (in ${ }^{\circ} \mathrm{F}$ ) at time $t$ in hours, and let $A=400$ be the ambient temperature of the oven. Let $t=0$ correspond to 10:00 am. We are given two pieces of information about the temperature of the turkey: $y(0)=50$ (the initial temperature of the turkey) and $y(2)=80$ (the temperature after two hours of cooking).
Using Newton's Law of Cooling, we have

$$
\frac{d y}{d t}=k(y-400)
$$

Using the Separation of Variables technique, we have

$$
\begin{aligned}
\frac{d y}{y-400}=k d t & \Longrightarrow \ln |y-400|=k t+c \\
& \Longrightarrow|y-400|=e^{k t+c}=c e^{k t} \\
& \Longrightarrow y-400=c e^{k t} \\
& \Longrightarrow y=400+c e^{k t} .
\end{aligned}
$$

Now we find the values of $c$ and $k$. Since $y(0)=50$, we have $50=400+c e^{0}=400+c$. Therefore $c=-350$. Then $y(2)=80$ implies $80=400-350 e^{2 k}$, which gives in turn

$$
\frac{-320}{-350}=e^{2 k} \quad \Longrightarrow \quad k=\frac{1}{2} \ln \left(\frac{32}{35}\right) \approx-0.044806
$$

Thus we have found the formula for the temperature, $y(t)=400-350 e^{-0.044806 t}$.
To find when the temperature reaches $150^{\circ} \mathrm{F}$, we set $y=150$ and solve for $t$. We have

$$
\begin{aligned}
150=400-350 e^{-0.044806 t} & \Longrightarrow \frac{-250}{-350}=e^{-0.044806 t} \\
& \Longrightarrow \ln \left(\frac{5}{7}\right)=-0.044806 t \\
& \Longrightarrow t=-\frac{1}{0.044806} \ln \left(\frac{5}{7}\right) \approx 7.5095 \text { hours. }
\end{aligned}
$$

Since $0.5095 \cdot 60=30.57 \approx 31$, the turkey is ready to serve in 7 hours and 31 minutes, which corresponds to $5: 31 \mathrm{pm}$. Auntie Pat better have some hors d'ouvres prepared!
9. Calculus potpourri:
(a) If $F(x)=\int_{x^{2}}^{5} \cos (\sqrt{t}+\pi) d t$, find $F^{\prime}(\pi)$.

Answer: $-2 \pi$. This is a problem using FTC, part 2 and the chain rule. First flip the limits of integration and then apply FTC, part 2 as well as the chain rule. We have

$$
\begin{aligned}
F^{\prime}(x) & =-\frac{d}{d x}\left(\int_{5}^{x^{2}} \cos (\sqrt{t}+\pi) d t\right) \\
& =-\cos \left(\sqrt{x^{2}}+\pi\right) \cdot \frac{d}{d x}\left(x^{2}\right) \\
& =-2 x \cos (x+\pi) .
\end{aligned}
$$

Then, since $F^{\prime}(x)=-2 x \cos (x+\pi)$, we have $F^{\prime}(\pi)=-2 \pi \cos (2 \pi)=-2 \pi$.
(b) Derive the formula for the volume of a sphere of radius $r$ by rotating the top half of the circle $x^{2}+y^{2}=r^{2}$ about the $x$-axis.

Answer: To find the volume of the sphere, we rotate the area under the curve $y=$ $f(x)=\sqrt{r^{2}-x^{2}}$ (semi-circle) about the $x$-axis from $x=-r$ to $x=r$. We use the disk
method since there is no gap between the region and the axis of rotation. Treating $r$ as a constant, the volume is given by

$$
\begin{aligned}
\pi \int_{-r}^{r}\left(\sqrt{r^{2}-x^{2}}\right)^{2} d x & =\pi \int_{-r}^{r} r^{2}-x^{2} d x \\
& =\pi\left(r^{2} x-\left.\frac{x^{3}}{3}\right|_{-r} ^{r}\right) \\
& =\pi\left(r^{3}-\frac{r^{3}}{3}-\left(-r^{3}+\frac{r^{3}}{3}\right)\right)=\frac{4}{3} \pi r^{3}
\end{aligned}
$$

(c) Find the average value of the function $g(x)=x \sin x$ over the interval $0 \leq x \leq \pi$.

Answer: 1. The average value of $f(x)$ over $[a, b]$ is $\frac{1}{b-a} \int_{a}^{b} f(x) d x$, so we need to calculate $\frac{1}{\pi} \int x \sin x d x$.
The integral is computed using integration by parts taking $u=x$ and $d v=\sin x d x$. Then we have $d u=1 d x$ and $v=-\cos x$. Applying the integration by parts formula, we find

$$
\begin{aligned}
\frac{1}{\pi} \int x \sin x d x & =\frac{1}{\pi}\left(-\left.x \cos x\right|_{0} ^{\pi}-\int_{0}^{\pi}-\cos x d x\right) \\
& =\frac{1}{\pi}\left(-\pi \cos \pi-0+\left.\sin x\right|_{0} ^{\pi}\right) \\
& =\frac{1}{\pi}(\pi+\sin \pi-\sin 0) \\
& =\frac{1}{\pi} \cdot \pi=1
\end{aligned}
$$

(d) Suppose that $p(x)$ is a piecewise function defined as follows:

$$
p(x)=\left\{\begin{array}{cc}
0 & \text { if } x<0 \text { or } x>2 \\
C x^{2}(2-x) & \text { if } 0 \leq x \leq 2
\end{array}\right.
$$

Find the value of $C$ which makes $p$ a probability density function.
Answer: $C=3 / 4$. Recall that a probability density function satisfies $\int_{-\infty}^{\infty} f(x) d x=1$. Since the function is zero for $x<0$ and $x>2$, we must solve

$$
\int_{0}^{2} C x^{2}(2-x) d x=1
$$

for $C$. We have

$$
\begin{aligned}
\int_{0}^{2} C x^{2}(2-x) d x & =C \int_{0}^{2} 2 x^{2}-x^{3} d x \\
& =C\left(\frac{2}{3} x^{3}-\left.\frac{x^{4}}{4}\right|_{0} ^{2}\right) \\
& =C\left(\frac{2}{3} \cdot 8-\frac{16}{4}-0\right) \\
& =C\left(\frac{16}{3}-4\right) \\
& =\frac{4 C}{3}
\end{aligned}
$$

Solving $4 C / 3=1$ gives $C=3 / 4$.
(e) Suppose that $p(x)$ is a probability density function and that $p(x)$ is an even function. If $\int_{2}^{\infty} p(x) d x=0.3$, what is $P(-2 \leq x \leq 2) ?$

Answer: 0.4. Recall that a probability density function satisfies $\int_{-\infty}^{\infty} p(x) d x=1$. Since $p$ is an even function, $\int_{-\infty}^{-2} p(x) d x=0.3$ by symmetry (equal areas over reflected intervals). Using linearity of the integral, we have

$$
\int_{-\infty}^{\infty} p(x) d x=\int_{-\infty}^{-2} p(x) d x+\int_{-2}^{2} p(x) d x+\int_{2}^{\infty} p(x) d x
$$

or

$$
1=0.3+\int_{-2}^{2} p(x) d x+0.3
$$

This implies that $P(-2 \leq x \leq 2)=1-0.6=0.4$.
(f) TRUE or FALSE: If true, provide a brief explanation. If false, give a counterexample to the statement.
If $\lim _{n \rightarrow \infty} a_{n}=0$, then the series $\sum_{n=1}^{\infty} a_{n}$ converges.
Answer: FALSE. The Harmonic Series is an excellent counterexample: $\sum_{n=1}^{\infty} \frac{1}{n}$. This series has the $n$th term approaching 0 , yet it still diverges (by the integral test or the proof given on the worksheet for Section 10.2). Other counterexamples include any $p$-series with $0<p<1$, such as

$$
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text { or } \quad \sum_{n=1}^{\infty} \frac{1}{n^{0.99}}
$$

10. Note: This question will be on the exam, so prepare your answer ahead of time. Pick one topic or idea from the course that you found interesting.
(a) Explain why this particular topic was interesting to you.
(b) What more would you like to learn about this topic?
(c) Pick a problem related to this topic and solve it.

Answer: I look forward to reading your response to this question.

