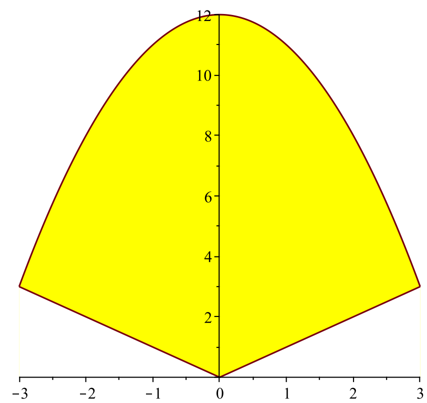
MATH 134 Calculus 2 with FUNdamentals

Practice Exam #2 SOLUTIONS

1. Let R be the region bounded by the curves $y = |x|$ and $y = 12 - x^2$.

   (a) Sketch the region R in the xy-plane.

   Answer:



To find where the two curves intersect, we first assume that $x > 0$ and then solve $x = 12 - x^2$. By assuming that $x > 0$, we can write $|x| = x$. Then, $x = 12 - x^2$ becomes $x^2 + x - 12 = 0$ or $(x+4)(x-3) = 0$. It follows that $x = 3$, since we assumed $x > 0$ (or just check that $x = -4$ gives a y-value of 4 when $y = |x|$, but -4 if $y = 12 - x^2$). By symmetry, the curves intersect at $x = 3$ and -3, as shown in the figure.

   (b) Find the area of the region R.

   Answer: 45

Using symmetry, we find the area of the region from $x = 0$ to $x = 3$ and then double the result. We have

$$\begin{aligned}
2\int_0^3 12 - x^2 - x\,dx &= 2\left(12x - \frac{x^3}{3} - \frac{x^2}{2}\Big|_0^3\right) \\
&= 2(36 - 9 - 9/2 - 0) \\
&= 2(27 - 9/2) \\
&= 54 - 9 \;=\; 45.
\end{aligned}$$

2. **Solids of Revolution:** Give the **exact** answers (no decimals).

   (a) Let A be the region under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \pi$. Find the volume of the solid of revolution obtained by rotating A about the x-axis.

   Answer: $\dfrac{\pi^2}{2}$

We use the disc method because there is no gap between the region and the axis of rotation. The radius is the height of the function, $r = \sin x$. The volume is given by

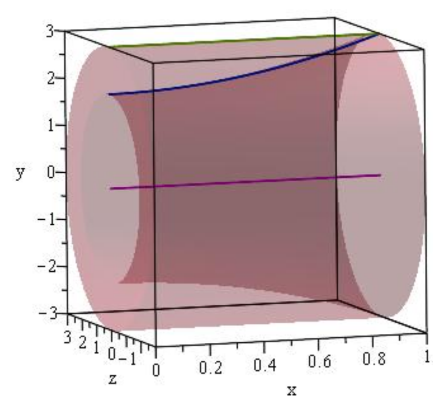
$$
\begin{aligned}
\int_0^{\pi} \pi(\sin x)^2\, dx &= \pi \int_0^{\pi} \sin^2 x\, dx \\
&= \pi \int_0^{\pi} \frac{1}{2}\left(1 - \cos(2x)\right)\, dx \\
&= \frac{\pi}{2}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_0^{\pi} \\
&= \frac{\pi}{2}\left(\pi - \frac{1}{2}\sin(2\pi) - 0 + \frac{1}{2}\sin(0)\right) \\
&= \frac{\pi^2}{2}.
\end{aligned}
$$

(b) Let B be the region enclosed by the graphs of $x = 0$, $y = 3$, and $y = x^2 + 2$. Find the volume of the solid of revolution obtained by rotating A about the x-axis.

Answer: $\dfrac{52\pi}{15}$

See the figure below. Using the washer method, the outer radius is 3 (distance between green line and violet axis) and the inner radius is $x^2 + 2$ (distance between blue parabola and violet axis). Solving $3 = x^2 + 2$ gives $x = 1$ as an intersection point. Thus, the volume is given by

$$
\begin{aligned}
\pi \int_0^1 3^2 - (x^2+2)^2\, dx &= \pi \int_0^1 9 - (x^4 + 4x^2 + 4)\, dx \\
&= \pi \int_0^1 5 - x^4 - 4x^2\, dx \\
&= \pi \left(5x - \frac{x^5}{5} - \frac{4x^3}{3}\Big|_0^1\right) \\
&= \pi\left(5 - \frac{1}{5} - \frac{4}{3}\right) = \frac{52\pi}{15}.
\end{aligned}
$$



3. Evaluate the following integrals using the appropriate method or combination of methods.

(a) $\displaystyle\int t^3 \ln t \, dt$

Answer: Use integration by parts. Let $u = \ln t$ and $dv = t^3\, dt$. This leads to $du = \frac{1}{t}\, dt$ and $v = \frac{1}{4}t^4$. The integration by parts formula then yields

$$\begin{aligned}
\int t^3 \ln t \, dt &= \frac{1}{4}t^4 \ln t - \int \frac{1}{4}t^4 \cdot \frac{1}{t}\, dt \\
&= \frac{1}{4}t^4 \ln t - \frac{1}{4}\int t^3\, dt \\
&= \frac{1}{4}t^4 \ln t - \frac{1}{16}t^4 + c \\
&= \frac{t^4}{16}(4\ln t - 1) + c\,.
\end{aligned}$$

(b) $\displaystyle\int \sin^5\theta \, d\theta$

Answer: The first step is to factor out $\sin\theta$ and use $\sin^2\theta = 1 - \cos^2\theta$. We have

$$\begin{aligned}
\int \sin^5\theta\, d\theta &= \int \sin\theta \cdot (\sin^2\theta)^2\, d\theta \\
&= \int \sin\theta \cdot (1-\cos^2\theta)^2\, d\theta \\
&= \int \sin\theta(1 - 2\cos^2\theta + \cos^4\theta)\, d\theta \\
&= -\int 1 - 2u^2 + u^4\, du \quad \text{using } u = \cos\theta, du = -\sin\theta\, d\theta \\
&= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + c \\
&= -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + c\,.
\end{aligned}$$

(c) $\displaystyle\int \frac{2x+24}{(x-3)(x+2)}\, dx$

Answer: Use partial fractions. We seek constants A and B such that

$$\frac{2x+24}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}\,.$$

Multiplying through by the LCD $(x-3)(x+2)$ gives

$$2x + 24 = A(x+2) + B(x-3).$$

Next we plug in the roots $x = -2$ and $x = 3$. Using $x = -2$ in the previous equation, we find $20 = -5B$ or $B = -4$. Likewise, setting $x = 3$ in the previous equation gives $30 = 5A$ or $A = 6$. Thus, the integral transforms into

$$\int \frac{6}{x-3} - \frac{4}{x+2}\, dx \;=\; 6\ln|x-3| - 4\ln|x+2| + c\,.$$

4. Evaluate the integral $\displaystyle\int \frac{1}{(9-x^2)^{3/2}}\, dx$ using the trig substitution $x = 3\sin\theta$.

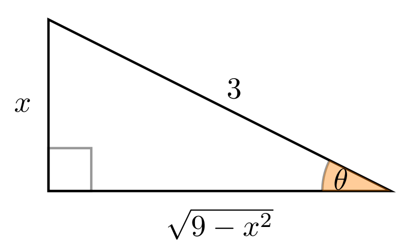
Answer: Letting $x = 3\sin\theta$, we have $dx = 3\cos\theta\, d\theta$ and $x^2 = 9\sin^2\theta$. Also, using the fundamental trig identity $\cos^2\theta + \sin^2\theta = 1$, the denominator simplifies to

$$(9 - 9\sin^2\theta)^{3/2} \;=\; (9(1-\sin^2\theta))^{3/2} \;=\; 9^{3/2}(\cos^2\theta)^{3/2} \;=\; 27\cos^3\theta\,.$$

We find

$$\begin{aligned}
\int \frac{1}{(9-x^2)^{3/2}}\, dx \;&=\; \int \frac{1}{27\cos^3\theta}\cdot 3\cos\theta\; d\theta \\
&=\; \frac{1}{9}\int \frac{1}{\cos^2\theta}\, d\theta \\
&=\; \frac{1}{9}\int \sec^2\theta\; d\theta \\
&=\; \frac{1}{9}\tan\theta + c \\
&=\; \frac{1}{9}\cdot\frac{x}{\sqrt{9-x^2}} + c \;=\; \frac{x}{9\sqrt{9-x^2}} + c\,.
\end{aligned}$$

The final step comes from using right-triangle trig and drawing a right triangle with opposite leg x and hypotenuse 3. The remaining leg is $\sqrt{9-x^2}$ by the Pythagorean theorem (see Figure below).



5. Consider the two integrals below. One of these can be found using a u-substitution while the other requires trig substitution. Determine which is which and evaluate **both** integrals.

(a) $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\, dx$ **(b)** $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\, dx$

Answer: The first integral can be evaluated using a u-substitution with $u = x^2 + 4$, while the second requires trig substitution.

For **(a)**, we let $u = x^2 + 4$ and then $du = 2x\, dx$ or $du/2 = x\, dx$. The integral becomes

$$\int \frac{1}{\sqrt{u}} \cdot \frac{du}{2} \;=\; \frac{1}{2}\int u^{-1/2}\, du \;=\; \frac{1}{2}\cdot 2u^{1/2} + c \;=\; \sqrt{x^2+4} + c\,.$$

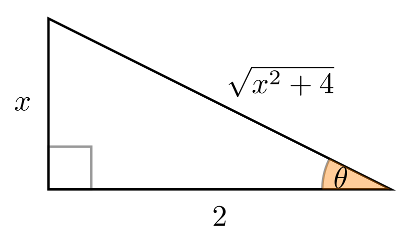
For **(b)**, let $x = 2\tan\theta$. Then we have $dx = 2\sec^2\theta\, d\theta$ and

$$x^2 + 4 \;=\; 4\tan^2\theta + 4 \;=\; 4(\tan^2\theta + 1) \;=\; 4\sec^2\theta.$$

Thus, the integral transforms to

$$\begin{aligned}
\int \frac{1}{\sqrt{4\sec^2\theta}} \cdot 2\sec^2\theta\, d\theta \;&=\; \int \frac{1}{2\sec\theta}\cdot 2\sec^2\theta\, d\theta \\
&=\; \int \sec\theta\, d\theta \\
&=\; \ln|\sec\theta + \tan\theta| + c \qquad \text{(\#14 on list of integral formulas)} \\
&=\; \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + c \qquad \text{(right-triangle trig; see Figure below)} \\
&=\; \ln\left|\frac{\sqrt{x^2+4} + x}{2}\right| + c\,.
\end{aligned}$$

Since $x = 2\tan\theta$, we have $\tan\theta = \dfrac{x}{2}$ and $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\sqrt{x^2+4}}{2}$.



6. **Calculus Potpourri:**

(a) Find the average value of $f(x) = xe^{3x}$ over the interval $[0,3]$. Give the **exact** answer (no decimals).

Answer: $\dfrac{8}{27}e^9 + \dfrac{1}{27}$. The average value of $f(x)$ over $[a,b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\, dx$, so we need to calculate $\dfrac{1}{3}\displaystyle\int xe^{3x}\, dx$.

The integral is computed using integration by parts taking $u = x$ and $dv = e^{3x}\, dx$. Then we have $du = 1\,dx$ and $v = \frac{1}{3}e^{3x}$. Applying the integration by parts formula, we find

$$\begin{aligned}
\frac{1}{3}\int xe^{3x}\,dx &= \frac{1}{3}\left(\frac{1}{3}xe^{3x}\Big|_0^3 - \int_0^3 \frac{1}{3}e^{3x}\,dx\right) \\
&= \frac{1}{3}\left(e^9 - 0 - \frac{1}{9}e^{3x}\Big|_0^3\right) \\
&= \frac{1}{3}\left(e^9 - \frac{1}{9}e^9 + \frac{1}{9}\right) \\
&= \frac{8}{27}e^9 + \frac{1}{27}.
\end{aligned}$$

(b) The population of Owenville has a radial density function of $\rho(r) = 20(3+r^2)^{-2}$, where r is the distance (in miles) from the city center and ρ is measured in thousands of people per square mile. Calculate the number of people living within 10 miles of the center of Owenville (round to the nearest whole number).

Answer: 20,334 people live within 10 miles of the center of Owenville.

The population is found by integrating the density function times $2\pi r$ over the interval $[0, 10]$. We have

$$\begin{aligned}
\int_0^{10} 2\pi r \cdot 20(3+r^2)^{-2}\,dr &= 20\pi \int_0^{10} 2r(3+r^2)^{-2}\,dr \\
&= 20\pi \cdot -(3+r^2)^{-1}\Big|_0^{10} \qquad (u\text{-sub with } u = 3+r^2) \\
&= 20\pi\left(\frac{-1}{103} + \frac{1}{3}\right) \\
&= \frac{2000\pi}{309} \\
&\approx 20.334 \text{ thousand people.}
\end{aligned}$$