MATH 134 Calculus 2 with FUNdamentals

Exam #3 SOLUTIONS **May 1, 2020** **Prof. Gareth Roberts**

1. **Quickies/Multiple Choice:** (5 pts. each)

(a) Suppose that $p(x)$ is a probability density function (PDF) and that $P(x \geq 3) = 0.75$ (using $p(x)$ as the PDF). What is the value of $\displaystyle\int_{-\infty}^{3} p(x)\,dx$?

(i) 0 **(ii)** 0.25 **(iii)** 0.5 **(iv)** 0.75 **(v)** diverges

Answer: **(ii)** 0.25. Recall that a probability density function satisfies $\displaystyle\int_{-\infty}^{\infty} p(x)\,dx = 1$. Since $P(x \geq 3) = 0.75$, we know that $\displaystyle\int_{3}^{\infty} p(x)\,dx = 0.75$. Using linearity of the integral, we have

$$\begin{aligned}
\int_{-\infty}^{3} p(x)\,dx &= \int_{-\infty}^{\infty} p(x)\,dx - \int_{3}^{\infty} p(x)\,dx \\
&= 1 - 0.75 \\
&= 0.25\,.
\end{aligned}$$

(b) Determine whether the sequence below converges or diverges. If it converges, state the limit.

$$a_n = \sqrt{17 - \frac{n}{n+1}}$$

Answer: The sequence converges to 4. This follows by first evaluating the limit inside the square root. First note that $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1$. Since the highest power in the numerator and denominator are the same, we just take the ratio of their coefficients (one could also use L'Hôpital's Rule). It follows that the sequence is approaching $\sqrt{17-1} = \sqrt{16} = 4$ as $n \to \infty$.

(c) Find the sum of the infinite series $25 - 10 + 4 - \dfrac{8}{5} + \dfrac{16}{25} - + \cdots$.

Answer: This is a geometric series with ratio $r = -2/5$ and starting term 25. The ratio can be found by inspection or by dividing any term by the term proceeding it (e.g., $-10 \div 25 = -2/5$, or $-\frac{8}{5} \div 4 = -2/5$). Then, using the formula for the sum of a geometric series, we have

$$S = \frac{a}{1-r} = \frac{25}{1-(-2/5)} = \frac{25}{\frac{7}{5}} = \frac{125}{7}\,.$$

(d) Give an example of an infinite series $\sum_{n=1}^{\infty} a_n$ satisfying $\lim_{n\to\infty} a_n = 0$, but for which the series **diverges**.

Answer: The Harmonic Series is an excellent example: $\sum_{n=1}^{\infty} \frac{1}{n}$. This series has the nth term approaching 0, yet it still diverges (by the integral test or the proof given on the worksheet for Section 10.2). Other examples include any p-series with $0 < p < 1$, such as

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n^{0.99}} .$$

2. **Compound Interest and Present Value:** (round all answers to the nearest cent) (16 pts.)

(a) Suppose that \$15,000 is invested in an account paying interest at an annual rate of 4% with the interest compounded **quarterly.** How much money will be in the account after 8 years?

Answer: Use the formula $P(t) = P_0\left(1 + \frac{r}{M}\right)^{Mt}$ with $P_0 = 15{,}000, r = 0.04, M = 4$, and $t = 8$. The amount of money in the account after 8 years is

$$P(10) = 15{,}000\left(1 + \frac{0.04}{4}\right)^{4\cdot 8} \approx \$20{,}624.11 .$$

(b) Is it better to receive \$1,000 today or \$1,400 in 5 years if the interest rate is 7%? Assume that interest is compounded continuously. Explain your answer.

Answer: It is better to receive \$1,000 today. To see this, we compute the Present Value PV of \$1,400 in 5 years with interest rate 7%. This tells us how much \$1,400 in 5 years is worth today. The present value is

$$\text{PV} = 1{,}400e^{-0.07\cdot 5} = 1{,}400e^{-0.35} \approx 986.56 .$$

Since \$986.56 < \$1,000, it is better to receive \$1,000 today.

(c) Compute the present value PV of an income stream paying out continuously at a rate of $R(t) =$ \$9,000 per year for 5 years, assuming an interest rate of 6%.

Answer: \$38,877.27. We have

$$\begin{aligned} PV &= \int_0^5 9000e^{-0.06t}\,dt \\ &= 9000 \cdot \frac{1}{-0.06}e^{-0.06t}\Big|_0^5 \\ &= -\frac{9000}{0.06}\left(e^{-0.3} - 1\right) \\ &\approx \$38{,}877.27. \end{aligned}$$

Notice that this is substantially less than \$45,000.00 (\$9,000 for 5 years), reflecting the loss of money caused by receiving payments over time rather than immediately.

3. **Improper Integrals:** Determine whether each improper integral converges or diverges. If the integral converges, give the exact value of the integral. (8 pts. each)

   (a) $\displaystyle\int_2^\infty \frac{1}{x \ln x}\, dx$

   Answer: The integral diverges. It can be computed with a u-substitution with $u = \ln x$, $du = \frac{1}{x}\, dx$. This transforms the integral into $\displaystyle\int \frac{1}{u}\, du = \ln|u|$. We have

$$\int_2^\infty \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} \ln|\ln x|\Big|_2^b = \lim_{b\to\infty} \ln|\ln b| - \ln|\ln 2| = \infty$$

   since $\displaystyle\lim_{x\to\infty} \ln x = \infty$. Therefore the integral diverges.

   (b) $\displaystyle\int_3^{12} \frac{1}{\sqrt{x-3}}\, dx$

   Answer: The integral converges to 6. It can be computed using a u-substitution with $u = x - 3$ and $du = dx$. Notice that the "bad" point is $x = 3$ since this value of x makes the denominator 0 (vertical asymptote). In the u variable, the limits of integration become $u = b - 3$ and $u = 9$. We have

$$\begin{aligned}
\int_3^{12} \frac{1}{\sqrt{x-3}}\, dx &= \lim_{b\to 3^+} \int_b^{12} (x-3)^{-1/2}\, dx \\
&= \lim_{b\to 3^+} \int_{b-3}^{9} u^{-1/2}\, du \qquad (u = x-3,\ du = dx) \\
&= \lim_{b\to 3^+} 2u^{1/2}\Big|_{b-3}^{9} \\
&= \lim_{b\to 3^+} 2\sqrt{9} - 2\sqrt{b-3} \\
&= 6 - 0 = 6\,.
\end{aligned}$$

4. **Probability:** Consider the piecewise function

$$p(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{2}{\pi} \cdot \dfrac{1}{x^2+1} & \text{if } x \geq 0 \end{cases}$$

   (a) Check that $p(x)$ is a probability density function, that is, show that $\displaystyle\int_{-\infty}^{\infty} p(x)\, dx = 1$. (8 pts.)

   Answer: Since $p(x) = 0$ for $-\infty < x < 0$, we need to show that $\displaystyle\int_0^\infty \frac{2}{\pi} \cdot \frac{1}{x^2+1}\, dx = 1$.

We have

$$\begin{aligned}
\int_0^\infty \frac{2}{\pi}\cdot\frac{1}{x^2+1}\,dx &= \lim_{b\to\infty}\int_0^b \frac{2}{\pi}\cdot\frac{1}{x^2+1}\,dx \\
&= \frac{2}{\pi}\lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx \\
&= \frac{2}{\pi}\lim_{b\to\infty}\tan^{-1}(x)\Big|_0^b \\
&= \frac{2}{\pi}\left(\lim_{b\to\infty}\tan^{-1}(b)-\tan^{-1}(0)\right) \\
&= \frac{2}{\pi}\left(\frac{\pi}{2}-0\right) \\
&= 1\,,
\end{aligned}$$

since $\lim_{x\to\infty}\tan^{-1}x = \pi/2$ (horizontal asymptote).

(b) Find $P(0 \le x \le 1)$. (5 pts.)

Answer: 1/2 or 50%. We compute

$$P(0\le x\le 1) = \int_0^1 \frac{2}{\pi}\cdot\frac{1}{x^2+1}\,dx = \frac{2}{\pi}\tan^{-1}(x)\Big|_0^1 = \frac{2}{\pi}\left(\tan^{-1}(1)-\tan^{-1}(0)\right) = \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2}\,.$$

5. **Infinite Series:** Determine whether the given infinite series converges or diverges using any of the tests discussed in class. You must state the test used and provide valid reasons to receive full credit. (15 pts.)

(a) $\displaystyle\sum_{n=1}^{\infty}\cos\left(\frac{1}{n}\right)$

Answer: This series diverges by the nth term test. Since $\displaystyle\lim_{n\to\infty}\frac{1}{n}=0$, we have

$$\lim_{n\to\infty} a_n \;=\; \cos(0) \;=\; 1 \;\neq\; 0$$

and thus the series diverges by the nth term test.

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt[3]{n}}$

Answer: This series diverges by the p-series test. Since $\sqrt[3]{n} = n^{1/3}$, the series is equivalent to $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^{1/3}}$, which is a p-series with $p = 1/3$. Since $1/3 < 1$, the series diverges by the p-series test.

(c) $\sum_{n=1}^{\infty} ne^{-n^2}$

Answer: This series converges by the integral test. The integral can be evaluated using a u-sub with $u = -x^2$ and $du = -2x\,dx$ or $x\,dx = -\frac{1}{2}\,du$. We have

$$\begin{aligned}
\int_1^{\infty} xe^{-x^2}\,dx &= \lim_{b\to\infty} \int_1^b xe^{-x^2}\,dx \\
&= \lim_{b\to\infty} \int_1^b e^u \cdot -\frac{1}{2}\,du \\
&= \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^b \\
&= \lim_{b\to\infty} -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1} \\
&= \lim_{b\to\infty} -\frac{1}{2e^{b^2}} + \frac{1}{2e} \\
&= \frac{1}{2e}\,.
\end{aligned}$$

Since the integral converges, the series also converges by the integral test.

6. **Calculus Potpourri:**

(a) Suppose that the demand curve for a box of nails is given by $p(x) = \dfrac{30}{x+2}$, where x is measured in thousands of boxes and p is given in dollars per box. If the supply curve is given by $s(x) = x + 3$ (same units), find the equilibrium price $\overline{p}$ and then compute the consumer surplus CS and producer surplus PS at the equilibrium price. (Round your answers to the nearest dollar.) (12 pts.)

Answer: The equilibrium price is $\overline{p} = \$6$, the consumer surplus is \$9,489, and the producer surplus is \$4,500.

To find the equilibrium price, we find where $p(x)$ and $s(x)$ intersect. Solving $p(x) = s(x)$ leads to the quadratic equation $x^2 + 5x - 24 = 0$. This factors as $(x+8)(x-3) = 0$, which means $\overline{x} = 3$ since $x = -8$ makes no sense physically. Plugging $x = 3$ into either $p(x)$ or $s(x)$ gives the equilibrium price of $\overline{p} = \$6$.
Next, using the integral formula for consumer surplus, we find that

$$\text{CS} = \int_0^3 \frac{30}{x+2} - 6\,dx = 30\ln|x+2| - 6x\big|_0^3 = 30\ln 5 - 18 - 30\ln 2 \approx 9.489 = \$9{,}489\,.$$

We multiply our answer by 1000 because x is measured in thousands of boxes and p is given in dollars per box. So the unit of area is thousands of dollars.
Using the integral formula for producer surplus, we find that

$$\text{PS} = \int_0^3 6 - (x+3)\,dx = \int_0^3 3 - x\,dx = 3x - \frac{1}{2}x^2\Big|_0^3 = 9 - \frac{9}{2} = \frac{9}{2} = \$4{,}500\,.$$

(b) A very bouncy ball is dropped from a height of 9 feet and begins to bounce vertically. Each time it strikes the ground, it returns to two-thirds (2/3) of its previous height. What is the total vertical distance traveled by the ball if it bounces **infinitely** often? (8 pts.)

Hint: Draw a picture and find the height of the first few bounces.

Answer: 45 feet. This one is tricky because we need to calculate the height of each bounce on the way down **and** on the way back up. The total distance can be computed as the sum of a geometric series and 9.

The ball starts at a height of 9 feet. The first bounce goes up a height of $9 \cdot 2/3 = 6$ feet, and then down 6 feet. The second bounce goes up a height of $6 \cdot 2/3 = 4$ feet, and then down 4 feet. The third bounce goes up a height of $4 \cdot 2/3 = 8/3$ feet, and then down $8/3$ feet, and so on. The total distance traveled by the ball is

$$\begin{aligned}
\text{Total distance} &= 9 + 2 \cdot 6 + 2 \cdot 4 + 2 \cdot \frac{8}{3} + 2 \cdot \frac{16}{9} + \cdots \\
&= 9 + 12 + 8 + \frac{16}{3} + \frac{32}{9} + \cdots \\
&= 9 + \frac{12}{1 - 2/3} \\
&= 9 + \frac{12}{\frac{1}{3}} \\
&= 9 + 36 \\
&= 45\,,
\end{aligned}$$

where we have summed an infinite geometric series with first term $a = 12$ and ratio $r = 2/3$.