MATH 133 Calculus 1 with FUNdamentals, Fall 2013

Handout with Exercises: Sections 1.1 and 1.2

The first chapter focuses on the major functions we will be studying throughout the semester. Many of these functions are excellent **models** for real-world phenomenon and are essential in the natural and social sciences. The key points of the first two sections are described in this handout. Much of this material is standard in pre-calculus courses. Please read the handout carefully and complete all the exercises.

1.1 Four Ways to Represent a Function

A **function** is a rule that assigns to each input element in the **domain** a unique output element in the **range**. The set of inputs to a function is called the **domain**, while the set of outputs is called the **range**. If a function $f : A \mapsto B$ maps from the set A to the set B (but not necessarily all of B), then we often call B the **co-domain**.

The four ways to represent a function referred to in the text are analytically (an explicit formula), graphically, numerically (table) and verbally (described in words). Typically, we will use x and t as the independent variables (inputs) and letters such as y, N, s (for position), v (for velocity) or a (for acceleration) as the dependent variables (outputs). When graphing a function, we will always assume the independent variable is plotted on the horizontal axis while the dependent variable is plotted on the vertical axis. In order to represent a function, a graph must pass the **vertical line test**, that is, any vertical line through the graph can only pass through at most one point. Otherwise, one input in the domain would have more than output in the range, violating the definition of a function.

We say that a function f is **increasing** on an interval I if

$$f(x_1) \ < \ f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

Likewise, f is **decreasing** on an interval I if

$$f(x_1) \ > \ f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

This is much easier to see visually. Increasing functions move upwards from left to right while decreasing functions move downwards from left to right.

A function that satisfies $f(-x) = f(x)$ for all x in its domain is called an **even** function. The graph of an even function is symmetric about the vertical axis. If a function satisfies $f(-x) = -f(x)$ for all x in its domain, then it is an **odd** function. The graph of an odd function is symmetric about the origin (after reflecting about both the horizontal and vertical axes, the same graph is obtained.)

Examples of even functions include $7, x^2, x^4, |x|, x^{-2}, \cos x$. (Note the **even** exponents on some of the examples.) Examples of odd functions include $x, x^3, x^5, 1/x, \sin x, \tan x$. (Note the **odd** exponents on some of the examples.)

Exercise 0.1 *Sketch the graph of an even function, an odd function and a function that is neither even nor odd. Can a function be* **both** *even and odd?*

1.2 Mathematical Models: A Catalog of Essential Functions

Linear $f(x) = mx + b$

The constant m represents the slope, while b is the y-intercept (where the line crosses the vertical axis). If x increases by one unit, then $f(x)$ increases (or decreases) by m units. If $m > 0$, then the line is increasing while if $m < 0$, the line is decreasing. When $m = 0$, the line has zero slope and is horizontal (a constant function). If (x_1, y_1) and (x_2, y_2) are two points on a line, then the slope m of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\Delta y}{\Delta x}.$$

Exponential $f(x) = ca^x$ or $N(t) = N_0 a^t$

Note that the variable x (or t) is an **exponent**, hence the name of the function. The constant a is called the **base** and should always be positive. If $a > 1$, then we have **exponential growth** while if $0 < a < 1$, then we have **exponential decay**. The constant c (or N_0) is arbitrary and represents the **initial population** in an exponential population model. The key difference between linear and exponential functions is that a constant change in x yields a constant change in y for a linear function, but a constant change in x yields a constant **ratio** in y for an exponential function. In other words, for an exponential function, if x increases by one unit, then $f(x)$ increases (or decreases) by a **factor** of a.

Piecewise Function

A function may be defined in different pieces by specifying which formula is to be used over a particular subset of the domain.

Exercise 0.2 *Sketch the graph of the piecewise function*

$$g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2 & \text{if } 0 \le x \le 3 \\ 4 - x & \text{if } x > 3. \end{cases}$$

Hint: For starters, over the domain $x < 0$ (the negative horizontal axis), you should draw the graph of x^2. Now draw the other two pieces of the function over the remaining two pieces of the domain.

Absolute Value $f(x) = |x|$

One particular piecewise function is critical in mathematics, the **absolute value** function. The graph of this function is a V with vertex at the origin. Although you may have learned that the absolute value is always positive, this hardly captures the meaning of this function. The absolute value is used to measure distance. For example, $|4| = 4$ and $|-4| = 4$ both indicate that the points 4 and -4 are each four units from 0 on the number line. The expression $|a - b|$ gives the distance between the numbers a and b on the number line. Thus, $|2 - 5| = 3$ since 2 and 5 are 3 units apart on the number line. Similarly, $|3 + 4| = |3 - -4| = 7$ since 3 and -4 are 7 units apart.

The piecewise definition for $|x|$ is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Polynomial $p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$

A **polynomial** is a sum of terms of the form ax^n where a is any constant and $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The degree of p is the largest power n in the expression. For example, $p(x) = 3x^2 + x - 4$ is a degree 2 polynomial or a **quadratic** function. Any quadratic has the form $p(x) = ax^2 + bx + c$ for some constants a, b, c. The function $p(x) = 5x^3 - x + \pi$ is a **cubic** polynomial (degree 3) and the function $p(x) = 17x + \sqrt{2}\,x^3 - 12x^4$ is a **quartic** polynomial (degree 4).

Rational $R(x) = p(x)/q(x)$

A rational function is simply the ratio of two polynomials. For example,

$$R(x) = \frac{x^2 - 1}{x^2 + 1} \qquad \text{and} \qquad S(x) = \frac{x^5 - 12x^3 + 6}{x^2 - 4}$$

are each rational functions.

Exercise 0.3 *Find the domain of each rational function $R(x)$ and $S(x)$ listed above.*

Power Function $f(x) = x^p, p \in \mathbb{R}$

The power function generalizes the monomials in a polynomial by allowing for any kind of exponent, not just natural numbers. Some examples of power functions include:

$$x^2, \ \sqrt{x} = x^{1/2}, \ x^{-1} = \frac{1}{x}, \ x^\pi, \ x^{\sqrt{2}}$$

You should know the graphs of $1/x$ and $\sqrt{x}$.

Exercise 0.4 *On the same axis, sketch the graphs of the functions* $1/x$ *and* $\sqrt{x}$*. At what point(s) do they intersect?*

Trig Functions $\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$

Trigonometric functions are extremely important. There is an excellent chart on page 2 of the course text (very front of the book – you might want to tear it out and use it as a handy reference chart!) as well as an informative Appendix C on trigonometry in the back of the book. When writing $\sin x$ or $\cos x$, it is *always* assumed that x is measured in **radians** not degrees. An angle of 1 radian is equivalent to the angle which cuts off 1 unit of arc length of the unit circle. This is approximately 57°. The idea is to lay a piece of string of length equal to the radius along the outside of the circle. The angle obtained is equal to one radian (note the similarities between the terms "radius" and "radian"). The key formula to remember is that

$$\pi \text{ radians} \;=\; 180^\circ.$$

This follows from the fact that the circumference of the unit circle (radius equals one) is 2π.

For a given angle θ, let l_θ represent the ray emanating from the origin that makes an angle of θ with the positive x-axis. It is important to remember that $\cos\theta$ equals the x-coordinate of the point of intersection between l_θ and the unit circle and that $\sin\theta$ equals the y-coordinate of this same point of intersection. Since the unit circle has the equation $x^2 + y^2 = 1$, we quickly have the important identity

$$\cos^2\theta + \sin^2\theta \;=\; 1.$$

Trig functions are called **periodic** functions because they repeat themselves after some time (called the period). The period of $\sin\theta$ and $\cos\theta$ is 2π.

The other trig functions are defined as follows:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}.$$

Exercise 0.5 *Evaluate each of the following expressions without using a calculator:* $\sin(\pi/2)$*,* $\cos(-\pi/2)$*,* $\cos(15\pi)$*,* $\tan(\pi)$*,* $\sec^2\theta - \tan^2\theta$*. What is the domain and range of* $\sin x$*? of* $\csc x$*?*