Topics in Mathematics: Math and Music
Comparing the Three Tuning Systems

Scale Degree	Solfège	Interval	Pythagorean	Just Intonation	Equal Temp.
1	Do	Uni.	1	1	1
2	Re	M2	$\frac{9}{8} = 1.125$	$\frac{9}{8} = 1.125$	$2^{2/12} \approx 1.1225$
3	Mi	M3	$\frac{81}{64} = 1.265625$	$\frac{5}{4} = 1.25$	$2^{4/12} \approx 1.2599$
4	Fa	P4	$\frac{4}{3} = 1.3\bar{3}$	$\frac{4}{3} = 1.3\bar{3}$	$2^{5/12} \approx 1.3348$
5	Sol	P5	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$	$2^{7/12} \approx 1.4983$
6	La	M6	$\frac{27}{16} = 1.6875$	$\frac{5}{3} = 1.6\bar{6}$	$2^{9/12} \approx 1.6818$
7	Ti	M7	$\frac{243}{128} = 1.8984375$	$\frac{15}{8} = 1.875$	$2^{11/12} \approx 1.8877$
$8 = 1$	Do	Oct.	2	2	2

Table 1: The ratios or multipliers used to **raise** a note (increase the frequency) by a given musical interval in the three different tuning systems: the Pythagorean scale, just intonation, and equal temperament. Pythagorean tuning and just intonation use rational numbers while equal temperament uses irrational multipliers (except for unison or the 2 : 1 octave).

Example 0.1 *Find the frequency of the note C$^\sharp$ above A440 Hz in each of the three different tuning systems.*

Solution: Since C$^\sharp$ is a major third above A440, we use the multipliers for scale degree 3 listed in Table 1. For the Pythagorean scale, we multiply 440 by 1.265625 to find that C$^\sharp$ is 556.875 Hz. Using just intonation, we should tune C$^\sharp$ to $440 \cdot 1.25 = 550$ Hz. Finally, in equal temperament, C$^\sharp$ is given by $440 \cdot 2^{4/12} \approx 554.365$ Hz. □

Example 0.2 *Find the frequency of the note F just below A440 Hz in each of the three different tuning systems.*

Solution: In this case, since the note F is a major third *below* A440, we must *divide* 440 by the multipliers for scale degree 3 listed in Table 1. For the Pythagorean scale, we calculate that the F just below A440 is $440 \div 1.1.265625 \approx 347.654$ Hz. Using just intonation, we tune the F just below A440 to $440 \div 1.25 = 352$ Hz. Finally, in equal temperament, the F just below A440 is $440 \div 2^{4/12} \approx 349.228$ Hz. □

Scale Degree	Solfège	Interval	Pythagorean	Just Intonation	Equal Temp.
1	Do	Uni.	0	0	0
2	Re	M2	203.9	203.9	200
3	Mi	M3	407.8	386.3	400
4	Fa	P4	498.0	498.0	500
5	Sol	P5	702.0	702.0	700
6	La	M6	905.9	884.4	900
7	Ti	M7	1109.8	1088.3	1100
$8 = 1$	Do	Oct.	1200	1200	1200

Table 2: A comparison of the three tuning systems using **cents**, rounded to one decimal place. Note that equal temperament does a good job approximating a perfect fifth (only 2 cents off), but is noticeably sharp (nearly 14 cents) of a just major third.

Cents were introduced by the mathematician Alexander Ellis (1804–90) around 1880. They are now a commonly used unit of measurement when comparing different tuning systems, or discussing non-traditional tunings. A typical listener can distinguish pitches that are between 4 and 8 cents apart. Cents are based on a logarithmic scale (like decibels). The formula for converting a ratio or multiplier r into cents is

$$\boxed{\#\text{ of cents} \ = \ 1200 \log_2(r) \ = \ 1200 \cdot \frac{\ln r}{\ln 2}\,.} \tag{1}$$

For example, a half step in equal temperament is given by the multiplier $2^{1/12}$. In cents, using the definition of the logarithm, this is

$$1200 \log_2(2^{1/12}) \ = \ 1200 \cdot \frac{1}{12} \ = \ 100 \text{ cents.}$$

Since all half steps are equal in equal temperament, one can easily obtain any interval in cents just by multiplying the number of half steps in the interval by 100 (see Table 2). For comparison, the Pythagorean comma is approximately 23.5 cents while the syntonic comma is roughly 21.5 cents.

Example 0.3 *Compute the number of cents in a minor third in each of the three different tuning systems.* **Note:** *The ratio to raise the pitch a minor third in Pythagorean tuning is* $32/27$ *(Exercise 6 in Section 4.1).*

Solution: In the Pythagorean scale, the ratio used to raise a pitch by a minor third is $r = 32/27$. To convert this ratio into cents, we apply equation (1) to find that the number of cents in a Pythagorean minor third is

$$1200 \cdot \frac{\ln(32/27)}{\ln 2} \ \approx \ 294.1 \text{ cents.}$$

In just intonation, the ratio for a minor third is $6/5$, so we have

$$1200 \cdot \frac{\ln(6/5)}{\ln 2} \ \approx \ 315.6 \text{ cents}$$

in a just minor third. Since a minor third has three half steps, the number of cents using equal temperament is simply $3 \cdot 100 = 300$. $\square$