MATH 110, Mathematics and Music, Spring 2007
The Monochord Lab: Length Versus Pitch

NAMES: ____

The goal of this lab project is for you to explore the relationship between the length of a string and the pitch sounded when the string is plucked. The device you will use to investigate this relationship is called a **monochord**. You will recreate a famous musical scale known as the Pythagorean Scale from 6th century BC. Once you have completed the first portion of the lab, I will check your work before passing out the second part of the lab.

You should work in a group of four to five people (one monochord per group) answering questions and filling in your results as you complete the lab. Please turn in one lab report per group, listing the names of all the members at the top of the first page. After the lab is complete, we will perform some simple pieces of music as a class. Before beginning the lab, please read the safety instructions below.

1 SAFETY

- The end of the string is sharp. Be careful when handling the monochord near the tuner end of the instrument.
- Over-tightening the string may cause it to break, allowing the two broken pieces to fly around somewhat. The broken ends will be sharp, so handle any broken string carefully. Keep your face away from the string so that you do not get hit in the eye by a breaking string. When re-stringing the instrument, wear safety glasses and keep your eyes at least one string-length (60 cm) away from the instrument.
- This instrument is not for "twanging" as rock guitarists play their instruments. This type of playing may generate a louder sound, but the string life is reduced significantly. Sufficient volume may be generated with just a gentle "pluck" of the string.

2 The Pythagorean Scale

As legend goes, the great Greek mathematician and philosopher Pythagoras was walking by a blacksmiths shop when he noticed that certain sounds of the hammers on the anvils sounded more consonant than others. Upon investigation, he discovered that the nicer sounds came from hammers whose weights were in simple proportions like $2:1$, $3:2$ and $4:3$. After further musical experiments with strings and lutes (see the woodcut from Franchino Gafurio 1492 on page 12 of the course text), he and his followers (the Pythagoreans) became convinced that basic musical harmony could be expressed through simple ratios of whole numbers. Their general belief was that the lower the numbers in the ratio, the better the notes sounded together. Thus, the natural building blocks of mathematics (small whole numbers) were aligned with the natural interval relationships used to create harmonious music! This helps explain the devotion of the Pythagoreans to rational numbers, numbers that can be expressed as the ratio of two integers.

Using the monochord, you will investigate this Pythagorean belief and find the ratios used in the oldest musical scale, **the Pythagorean Scale**. The goal is to investigate the relationship between the length of the string vibrating and the pitch produced.

2.1 Part I: Simple Ratios

1. You can change the tension T of the string by tightening the lever at the right-end of the monochord. What happens to the pitch as you tighten the string?

2. Place the sliding fret exactly at 30 cm. Hold one finger over the fret and pluck the string gently. What is the relationship between the pitch of this note versus the note produced without the fret? If you are not sure about the relationship, you can try finding the notes on the piano although you will first have to "tune" the monochord (on the open, unfretted string) to a note on the piano.

3. Place the sliding fret exactly at 15 cm, hold the string down over the fret and pluck the smaller portion of the string (so you are vibrating a string of length 15 cm.) What is the pitch relationship between this tone and the one obtained by plucking a string of length 30 cm? What is the relationship between the 15 cm pitch and the open plucked string (60 cm)?

4. Place the sliding fret exactly at 20 cm, hold the string down over the fret and pluck each string (20 cm and 40 cm). What is the relationship between these two pitches? Draw an important conclusion:

   By cutting the length of the string in **half**, we ____________ (raise or lower) the pitch by ____________ (what interval)?

5. Next, investigate what happens if the ratio of the string lengths is 2 : 3. In other words, changing the string length to 2/3 its original value does what to the pitch? Be sure to pluck the correct side of the string. *Hint:* What is two-thirds of 60?

6. Next, investigate what happens if the ratio of the string lengths is 3 : 4. In other words, changing the string length to 3/4 its original value does what to the pitch? Again, be sure to pluck the correct side of the string. Given that $\frac{3}{4} = \frac{1}{2} \cdot \frac{3}{2}$, how could you have predicted this answer from the previous two questions?

2.2 Part 2: Ratios for the Pythagorean Scale

Given the ground work we've done above, you are ready to construct the entire Pythagorean scale. However, although it sounds like the usual diatonic major scale, it differs in some important ways from the scale on the piano. We will learn more about this later when we study equal temperament as a tuning system.

Assuming that you correctly completed Part 1 of the lab, you have discovered three important facts:

1. String lengths in a ratio of 1 : 2 are an octave apart. Specifically, cutting the length of the string in half, raises the pitch an octave. Conversely, doubling the length of the string, lowers the pitch an octave.

2. String lengths in a ratio of 2 : 3 are a perfect fifth apart. Specifically, cutting the length of the string by two-thirds, raises the pitch a perfect fifth.

3. String lengths in a ratio of 3 : 4 are a perfect fourth apart. Specifically, cutting the length of the string by three-quarters, raises the pitch a perfect fourth.

Note the musical simplicity above. Simple ratios lead to the "perfect" intervals of an octave, fifth and fourth. It is no accident that these are precisely the same intervals which composers began using hundreds of years ago (eg. Gregorian chant used octaves, Medieval polyphony used fourths and fifths.) It is also no surprise that these are the standard intervals used in popular music (I - IV - V) since they "sound nice." We will see why these intervals are so pleasing when we study the overtone series.

It is also important to note that Fact 3 above is superfluous in the sense that it follows from Facts 1 and 2. Suppose that the open string at 60cm sounds middle C on the piano. Then by Fact 1, a string of length 30 cm will sound the C an octave higher than middle C, call it C'. By Fact 2, a string of length 40 cm will sound the G above middle C (a perfect fifth higher.) Now, the interval between G and C' (G being the lower note) is a perfect fourth. Since the ratio between the two strings sounding G and C' is 30 : 40 or 3 : 4, we have derived Fact 3 from Facts 1 and 2. Alternatively, $30 = 3/4 \cdot 40$ so cutting the G string by a factor of 3/4 makes it 30 cm long, the length for C', and raises the pitch a perfect fourth.

The goal for this section of the lab is to fill out the remaining ratios in the Pythagorean scale. The scale is essentially the same as the major scale except the string lengths are slightly different from the modern ones. We can derive this scale using only Facts 1 and 2 above in a fashion similar to the *Circle of Fifths*. For example, to find the second note of the major scale, we go up two perfect fifths from the tonic and then down an octave. If the starting note is middle C, then up a perfect fifth gives G and then up another perfect fifth gives D'. We then go down an octave to get the note D. Using the ratios given in Facts 1 and 2, what fraction of the full string do we take to obtain the second note of the scale? Use a calculator to find the actual length of the string (round to two decimal places) and play it with the open string to hear the first two notes of the scale (Do and Re).

The next note to find would be the sixth of the scale since that is a perfect fifth above the second. From there we can find the third and finally the seventh note of the scale, the leading tone. Fill out the table below as you find the lengths and ratios of the string. When you have found the complete scale, play it and see if you can hear any differences with the scale on the piano.

Scale Degree	Solfege Syllable	Interval	Ratio	Length in cm
1	Do	unison	$\frac{1}{1} \cdot 60$	60
2	Re	major second		
3	Mi	major third		
4	Fa	perfect fourth	$\frac{3}{4} \cdot 60$	45
5	Sol	perfect fifth	$\frac{2}{3} \cdot 60$	40
6	La	major sixth		
7	Ti	major seventh		
1	Do	octave	$\frac{1}{2} \cdot 60$	30

Table 1: The lengths and ratios of the Pythagorean scale. Give lengths to two decimal places.

2.3 Some Concluding Questions

1. Now that you have completed the table, do you notice anything interesting about the ratios in column 4? Try writing the prime factorization for the numerator and denominator for each ratio. For example, $3/4 = 3/2^2$.

2. Using your table, what is the ratio of string lengths which are a whole-step apart? You should check **all** five whole-steps in the major scale to see that the ratio is identical for each whole step. This ratio, call it w, is the factor you would multiply the length by to raise the pitch a whole-step. Express w as a ratio of two integers.

3. Using your table, what is the ratio of string lengths which are a half-step apart? You should check **both** half-steps in the major scale. This ratio, call it h, is the factor you would multiply the length by to raise the pitch a half-step. Express h as a ratio of two integers (you might choose to use the prime factorization of the numerator and denominator.)

4. Knowing that two half-steps is equivalent to a whole step, it should be true that $h^2 = w$. Note that h is squared because we must *multiply* twice to raise the pitch two half-steps. Compute the value of h^2/w, giving your answer as a ratio of two integers and in decimal form (5 decimal places). The fact that this value is not equal to one is a problem! If two half-steps do not equal one whole step, how can we expect to play in a different key? How do we sharp or flat a note? This is why the Pythagorean scale is not commonly used. The value of h^2/w is called the **Pythagorean Comma**.