Mathematics and Music: Group Theory

Recall: The set G is a **group** under the operation $*$ if the following four properties are satisfied:

1. Closure: If $a \in G$ and $b \in G$, then $a * b \in G$. This must be true for all elements a and b in the group G.
2. Associativity: $(a * b) * c = a * (b * c)$
3. Identity: There must exist an element $e \in G$ called the **identity element** such that $a * e = a$ and $e * a = a$. (e preserves the "identity" of the element it is being multiplied by.)
4. Inverse: For every element $a \in G$, there must exist an element $a^{-1} \in G$ called the **inverse of** a, such that $a * a^{-1} = e$ and $a^{-1} * a = e$. Note that the inverse of each element must be in the group G.

Recall the extent on 4 bells called Plain Bob Minimus.

$e =$	**1** 2 3 4	**1** 3 4 2	**1** 4 2 3
$\alpha =$	2 **1** 4 3	3 **1** 2 4	4 **1** 3 2
$\beta =$	2 4 **1** 3	3 2 **1** 4	4 3 **1** 2
$\alpha\beta =$	4 2 3 **1**	2 3 4 **1**	3 4 2 **1**
$\beta^2 =$	4 3 2 **1**	2 4 3 **1**	3 2 4 **1**
$\alpha\beta^2 =$	3 4 **1** 2	4 2 **1** 3	2 3 **1** 4
$\beta^3 =$	3 **1** 4 2	4 **1** 2 3	2 **1** 3 4
$\beta\alpha =$	**1** 3 2 4	**1** 4 3 2	**1** 2 4 3
			1 2 3 4

We label the first two changes after rounds as α and β (α and β are pronounced "alpha" and "beta" respectively. These are the first two letters of the Greek alphabet, commonly used in mathematics.) One can check that the remaining 5 changes of the first lead are all expressable in terms of α and β, given by the formulas shown above. Note that we are dropping the $*$ here for ease of notation, so for example, $\alpha\beta = \alpha * \beta$. Remember that $\alpha\beta$ means we apply the permutation α first, then take the result and apply β. The order often matters!

On HW#7, one of the goals is to prove that the first column (lead) of Plain Bob Minimus forms a group under multiplication of permutations (Set A). This group is a **subgroup** of the larger group S_4. The hardest thing to show is that the elements in Set A are closed under multiplication of permutations. In other words, the product of any two permutations in the first column gives you a permutation that is still in the first column. This can be accomplished by making a multiplication table and checking that all 64 products are indeed elements of the first lead.

Instead of doing every multiplication out by hand, there are some identities involving α and β which are particularly useful. In turn, these identities can be used to derive other useful relations. You might keep a list of them as you fill out your table. Here are the key ones:

$$\alpha^2 = e$$
$$\beta^4 = e$$
$$\beta\alpha\beta = \alpha$$

Using the identities: Suppose we wanted to find the product $\alpha\beta^3$ using the identities above. Multiplying **on the right** of each side of the last identity gives us

$$\beta\alpha\beta * \beta^3 = \alpha * \beta^3$$

The left-hand side of this equation simplifies to $\beta\alpha\beta^4 = \beta\alpha e = \beta\alpha$ which is listed in the fifth row of Table 1. It is very important that you multiply the same way on each side of the equation. Since $*$ is not usually commutative, the order matters! The goal is for you to fill out the multiplication table below, **obtaining only elements from our supposed subgroup Set A.**

$*$	e	β	β^2	β^3	α	$\alpha\beta^2$	$\alpha\beta$	$\beta\alpha$
e	e	β	β^2	β^3	α	$\alpha\beta^2$	$\alpha\beta$	$\beta\alpha$
β								
β^2								
β^3								
α	α	$\alpha\beta$	$\alpha\beta^2$	$\beta\alpha$	e	β^2	β	β^3
$\alpha\beta^2$								
$\alpha\beta$								
$\beta\alpha$								

Table 1: Multiplication table for the 8 permutations in the first lead of Plain Bob Minimus. Fill this out for HW#7, question 7. Two rows have already been completed to help you get started.

Mathematical side note: It is tempting to ask whether the other leads in Plain Bob Minimus are also subgroups. The answer is no since neither contains rounds, which is the identity element, so property 3 for groups does not hold. However, it is easy to generate these leads from the subgroup formed by the first lead. Multiplication by the permutation (1 3 4 2) on the right takes the entire first column to the second column and multiplication again by this same permutation takes the second column to the third. In group theory, the second and third leads are called **cosets**. A coset is obtained from a subgroup by multiplying on the left or the right every element in the subgroup. Thus we can speak of left or right cosets. Each coset has the same number of elements as the subgroup just as our three leads each have the same number of changes. It turns out that many extents have this decomposition where the first lead is a subgroup and the remaining leads are just cosets generated by this group. Moreover, the elements used in generating the other leads form a different subgroup together, called a **cyclic subgroup of order** n. This is a group of the form $\{\omega, \omega^2, \omega^3, \ldots, \omega^n = e\}$. It is clear that there is a great deal of group theory involved in doing change ringing.

Symmetries of the Square:

$*$	e	R_{90}	R_{180}	R_{270}	H	V	D_{13}	D_{24}
e	e	R_{90}	R_{180}	R_{270}	H	V	D_{13}	D_{24}
R_{90}								
R_{180}								
R_{270}								
H	H	D_{13}	V	D_{24}	e	R_{180}	R_{90}	R_{270}
V								
D_{13}								
D_{24}								

Table 2: Multiplication table for the 8 symmetries of the square. Fill this out for HW#7, question 9. Two rows have already been completed for you. Compare this table to Table 1. What similarities do you notice?