Mathematics and Music: Change Ringing

Rules to ring an extent on n bells:

1. The first and last changes (rows) are rounds ($1\ 2\ 3\ 4 \cdots n$).
2. Other than rounds, all of the other $n!$ changes occur exactly once.
3. Between successive changes, no bell moves more than one position.
4. No bell rests for more than 2 (sometimes relaxed further to 4) positions.
5. Each working bell should do the same amount of "work."
6. Horizontal symmetry should be present in the extent to help the ringers learn the path of their respective bell.

Note that Rules 1 - 3 are mandatory for an extent while Rules 4 - 6 are optional though often satisfied.

n	$n!$	Approximate Duration	Name
3	6	15 secs.	*Singles*
4	24	1 mins.	*Minimus*
5	120	5 mins.	*Doubles*
6	720	30 mins.	*Minor*
7	5,040	3 hrs.	*Triples*
8	40,320	24 hrs.	*Major*
9	362,880	9 days	*Caters*
10	3,628,800	3 months	*Royal*
11	39,916,800	3 years	*Cinques*
12	479,001,600	36 years	*Maximus*

Table 1: Approximate duration to ring an extent on n bells and the names given to such an extent. For example, *Plain Bob Minimus* is a composition written for 4 bells while *Grandshire Triples* is an extent on 7 bells.

The two extents on 3 bells:

1 2 3	**1** 2 3
2 **1** 3	**1** 3 2
2 3 **1**	3 **1** 2
3 2 **1**	3 2 **1**
3 **1** 2	2 3 **1**
1 3 2	2 **1** 3
1 2 3	**1** 2 3

Note the simple zig-zag pattern of Bell **1** in the first extent, sweeping easily from position 1 to position 3 and back again. We say that Bell **1** is *plain hunting*. It only needs to do this once to complete the extent. In this case, we say that the bell is "not working." Notice that in the second extent, Bell **1** follows a similar zig-zag path except that this begins on the second change.

Plain Bob Minimus: (read down first, then hop to next column)

1 2 3 4	**1** 3 4 2	**1** 4 2 3
2 **1** 4 3	3 **1** 2 4	4 **1** 3 2
2 4 **1** 3	3 2 **1** 4	4 3 **1** 2
4 2 3 **1**	2 3 4 **1**	3 4 2 **1**
4 3 2 **1**	2 4 3 **1**	3 2 4 **1**
3 4 **1** 2	4 2 **1** 3	2 3 **1** 4
3 **1** 4 2	4 **1** 2 3	2 **1** 3 4
1 3 2 4	**1** 4 3 2	**1** 2 4 3
		1 2 3 4

Note the similarities with the first extent on three bells above. Bell **1** goes plain hunting again, this time moving from position 1 to position 4 and back again, needing 3 cycles to complete the extent. The other three bells have the same paths, just starting at different places so that Rule 5 is satisfied. This is a bit like a round. There are lots of interesting patterns and mathematics lurking in this extent!

Canterbury Minimus: (read down first, then hop to next column)

1 2 3 4	**1** 3 4 2	**1** 4 2 3
2 **1** 4 3	3 **1** 2 4	4 **1** 3 2
2 4 **1** 3	3 2 **1** 4	4 3 **1** 2
2 4 3 **1**	3 2 4 **1**	4 3 2 **1**
4 2 3 **1**	2 3 4 **1**	3 4 2 **1**
4 2 **1** 3	2 3 **1** 4	3 4 **1** 2
4 **1** 2 3	2 **1** 3 4	3 **1** 4 2
1 4 3 2	**1** 2 4 3	**1** 3 2 4
		1 2 3 4

What are the similarities and differences with Plain Bob Minimus? Is this a legitimate extent? Which of the six rules does it satisfy?