MATH 110 Exam 2 Solutions
Mathematics and Music

April 12, 2006 Prof. G. Roberts

1. **Fill in the blanks:** Work is **required** to receive partial credit. (4 pts. each)

   (a) In the overtone series, the musical interval between the 4th and 5th harmonics, that is, between $4f$ and $5f$, is <u>a major 3rd</u>.

   This follows because the ratio of $5f$ to $4f$ is $5/4$ which is a major third in just intonation (approximately $81/64$ in Pythagorean tuning.)

   (b) In change ringing, a legitimate extent on n bells always ends with the permutation <u>(1 2 3 . . . n)</u>.

   (c) In change ringing, the number of admissible permutations (legal moves) to apply to 6 bells is <u>12</u>. (eg. (1 2) is a legal move but (1 3) is not.)

   This was a homework problem. The table of allowable moves begins 1, 2, 4, 7, 12, 20, etc.

   (d) The number of possible ways to ring 10 bells such that Bell 3, Bell 6 and Bell 9 are each fixed in positions 3, 6 and 9 respectively, is <u>$7! = 5040$</u>.

   (e) If the permutation $a = (2\ 3\ 4\ 1\ 6\ 5)$, then $a^3 =$ <u>$(4\ 1\ 2\ 3\ 6\ 5)$</u>.

   Using the rule for multiplying permutations, we have $a^2 = (3\ 4\ 1\ 2\ 5\ 6)$ and then $a^3 = a \cdot a^2 = (4\ 1\ 2\ 3\ 6\ 5)$.

   (f) Suppose that b is an element of a group G and that $b^5 = e$. Then $b^{101} =$ <u>b</u>.

   This follows since $101 = 5 \cdot 20 + 1$ so that $b^{101} = b \cdot b^{100} = b \cdot e = b$.

2. **Tuning Systems**

   (a) Given that A above middle C has a frequency of 440 Hz, find the frequency for D above middle C using Just Intonation. (4 pts.)

   Answer: $293.\bar{3}$ Since the interval from D up to A is a perfect fifth, we take 440 Hz and divide by $3/2$. This gives $440 \times 2/3 = 880/3 = 293.\bar{3}$. Alternatively, dropping an octave gives the A below D at 220 Hz. Then raising the pitch up a perfect fourth is equivalent to multiplying by $4/3$ giving $220 \times 4/3 = 880/3 = 293.\bar{3}$.

   (b) Give two advantages of using Just Intonation as a tuning system. (4 pts.)

   Answer: Just Intonation is useful for playing in one key (assuming you have tuned the tonic to that key) and particularly nice when playing the major chord of the tonic (1, 3 and 5 of the scale.) Because the tuning used in Just Intonation matches the overtone series, this chord will sound particularly harmonious. Just Intonation also uses ratios with small integers (again coming from the overtone series) and these are easy to remember.

   (c) Give two advantages of using Equal Temperament as a tuning system. (4 pts.)

   Answer: Equal Temperament solves the issue of the spiraling circle of fifths since 7 octaves in ET is equivalent to 12 perfect fifths (B$\sharp$ = C). Mathematically, this is $(2^{7/12})^{12} = 2^7$. Two half steps equals one whole step in ET and since the octave is equally divided into 12 pieces, it is now possible to switch keys easily using ET.

3. **Irrational and Rational Numbers**

(a) Using Equal Temperament, what number do you multiply the fundamental frequency of a note by to raise it up a perfect fifth? Give the **exact** value. (4 pts.)

Answer: $2^{7/12}$ since a perfect fifth is 7 half-steps and one half-step in Equal Temperament is obtained by multiplying by $2^{1/12}$.

(b) Prove that your answer to part **(a)** is an irrational number. (8 pts.)

Answer: Suppose by contradiction that $2^{7/12}$ is a rational number. Then we have $2^{7/12} = p/q$ for some integers p and q. Raising both sides to the 12th power gives $2^7 = p^{12}/q^{12}$. Cross-multiplying gives

$$2^7 \cdot q^{12} \; = \; p^{12}$$

which contradicts the Fundamental Theorem of Arithmetic because the left-hand side will have a prime factorization with an odd number of 2's (7 plus even = odd) while the right-hand side will have an even number. Therefore, our original assumption of rationality has produced a contradiction and $2^{7/12}$ must be an irrational number.

4. **Change Ringing:** Below is a list of 25 changes to be rung on 4 bells (read from top to bottom, then hop to the next column.) Let $a = (1\ 2)(3\ 4), b = (2\ 3), c = (3\ 4), d = (1\ 2)$ represent the four possible permutations of four bells.

1 2 3 4	1 3 4 2	1 4 2 3
2 1 3 4	3 1 4 2	4 1 2 3
2 3 1 4	3 4 1 2	4 2 1 3
2 3 4 1	3 4 2 1	4 2 3 1
3 2 4 1	4 3 2 1	2 4 3 1
3 2 1 4	4 3 1 2	2 4 1 3
3 1 2 4	4 1 3 2	2 1 4 3
1 3 2 4	1 4 3 2	1 2 4 3
		1 2 3 4

(a) Which of the bells (if any) go *plain hunting*? (4 pts.)

Answer: Bell 1 is the only bell which is plain hunting (zig-zag pattern.)

(b) Write the 25 changes above in terms of the permutations (a, b, c, d) which create it. (6 pts.)

Answer: $(d\,b\,c\,d\,c\,b\,d\,c)^3$

(c) Which of the six rules for an extent are satisfied? Do the changes above form a legitimate extent? (8 pts.)

Answer: Rules 1, 2, 3, 5 and 6 are satisfied. The extent starts and ends on rounds, has no change repeating except the first and last, covers all 24 changes in S_4, has only allowable moves (a, b, c, d), has all working bells (Bells 2, 3 and 4) doing the same amount of work (note that Bell 3 follows the path of Bell 2 and Bell 4 follows the path of Bell 3) and has the palindrome property.

Rule 4 is not satisfied in the strict sense because the working bells often sit in the same place three changes in a row. Because the first three rules are satisfied, this is a legitimate extent.

5. Recall that D_4, the dihedral group of degree 4 (symmetries of the square), contains the 8 elements $\{e, R_{90}, R_{180}, R_{270}, H, V, D_{13}, D_{24}\}$. The operation $*$ of the group is given by composition of transformations. For the questions below let

$$A = \{e, H, V\} \quad \text{and} \quad B = \{e, H, V, R_{180}\}$$

be two subsets of D_4. You may assume that $*$ is associative.

(a) Explain why the subset A is **not** a subgroup of D_4. (4 pts.)

Answer: There are two possible answers. For one, the set A is not closed under multiplication since $H * V = R_{180}$ which is not in A. Alternatively, the set A has only three elements but any subgroup of a finite group has a number of elements which is a factor of the order of G. Since 3 is not a factor of 8, A can not possibly be a subgroup of D_4.

(b) Complete the multiplication table below for the elements in the subset B. (6 pts.)

$*$	e	H	V	R_{180}
e	e	H	V	R_{180}
H	H	e	R_{180}	V
V	V	R_{180}	e	H
R_{180}	R_{180}	V	H	e

(c) Give the inverses of each element in B. In other words, fill in the following: (4 pts.)

$$e^{-1} = \underline{e} \qquad H^{-1} = \underline{H} \qquad V^{-1} = \underline{V} \qquad R_{180}^{-1} = \underline{R_{180}}$$

(d) Is the subset B a subgroup of D_4? Is it commutative? Explain. (6 pts.)

Answer: Yes, B is a subgroup of D_4. It is closed as can be seen from the multiplication table above. It contains the identity element e and each element has an inverse in B by our answer to **(c)**. Associativity of $*$ was given so all four properties of a group are satisfied. This is a commutative subgroup because the multiplication table is symmetric about the diagonal of e's. In other words, $a * b = b * a$ for all a, b in B.

6. **Musical Group Theory:** Answer questions **(a)** and **(b)** based on the following motif.

(a) Write the motif above after applying an inversion about the note B, staying in the given key. (5 pts.)

Answer: Inverting about the note B (middle of the staff) means we should start on the note G (second line from bottom.) Continuing on we have the notes F♯, A, B, D, C, D, with these notes getting the same rhythmic value as their counterparts in the original.

(b) Write the motif above after applying a retrograde-inversion, staying in the given key. Do the inversion about the note B. (5 pts.)

Answer: Take the retrograde of your answer to **(a)** (write it backwards).

(c) Give two pieces (include composer and title) that make use of symmetry under a vertical reflection. (4 pts.)

Answer: Handel's *Messiah*, Bach's Crab Cannon in *A Musical Offering*, Haydn's *Piano Sonata No. 41*, Gershwin's *I Got Rhythm* and *Lean on Me* by Bill Withers (1972) are all possible choices.