

① a)  $\frac{dy}{dx} = 2x \cdot \tan(\sin x) + x^2 \cdot \sec^2(\sin x) \cdot \cos x$   
 $= x (2 \tan(\sin x) + x \cos x \cdot \sec^2(\sin x))$

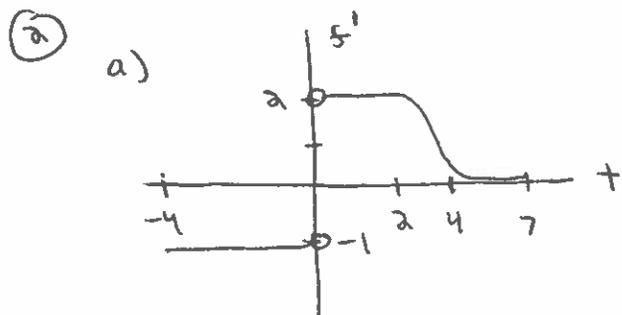
b)  $y = (x^4 + 10)^{-1/2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2} (x^4 + 10)^{-3/2} \cdot 4x^3 = -2x^3 (x^4 + 10)^{-3/2}$

c)  $\frac{dy}{dx} = 3x^2 + \ln 3 \cdot 3^x$

d) Use logarithmic differentiation:  $\ln y = \ln x^{x^2} = x^2 \ln x$

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 2x \cdot \ln x + x^2 \cdot \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = y (2x \cdot \ln x + x) = x^{x^2} \cdot x (2 \ln x + 1)$



b)  $F(-2) = \int_{-4}^{-2} f(t) dt = 2$   
 (area under curve)

$F(0) = \int_{-4}^0 f(t) dt = 0$

c) By FTC part II,  $F'(x) = f(x) \Rightarrow F''(x) = f'(x)$

$\Rightarrow F''(0) = f'(0)$  which doesn't exist since there is a corner in the graph of  $f$  at  $t=0$ .

d)  $F(-4) = 0$

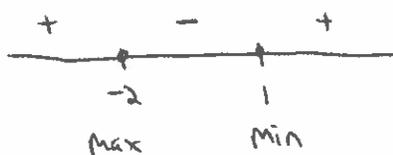
$F(-2) = 2$

$F(0) = 0$

$F(1) = -1$

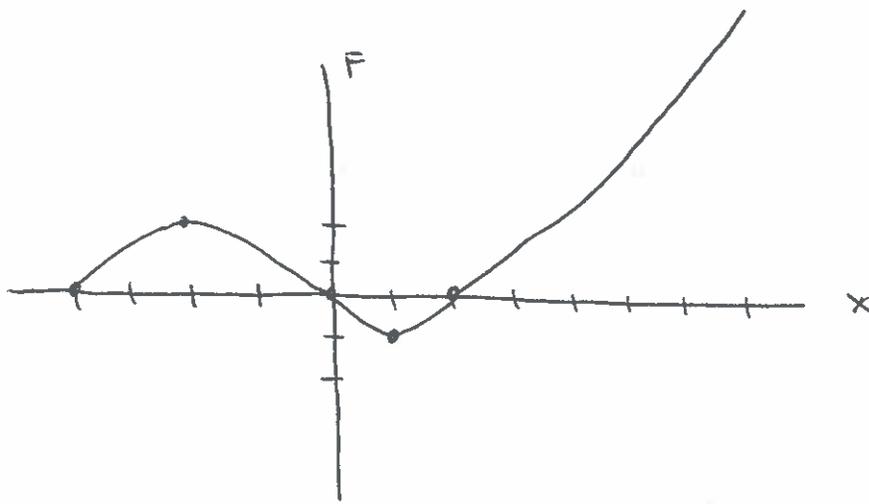
$F(2) = 0$

1<sup>st</sup> deriv.  
# line



The graph shown,  $f$ , is the graph of the derivative  $F'$ .

d)



e) Since  $f$  has a horizontal asymptote at  $y=3$ ,

$$F' \rightarrow 3 \text{ as } x \rightarrow \infty \Rightarrow F(x) \approx 3x + c \text{ as } x \rightarrow \infty.$$

ie.  $F$  resembles a linear function of slope 3 as  $x \rightarrow \infty$ .

③ (a) Since  $x^2+4 \neq 0$ , there are no vertical asymptotes.

Since  $\lim_{x \rightarrow \infty} f(x) = 0$ ,  $y=0$  is a horizontal asymptote.

$$b) f'(x) = \frac{(x^2+4) \cdot 3 - 3x \cdot 2x}{(x^2+4)^2} = \frac{-3x^2 + 12}{(x^2+4)^2} = \frac{-3(x^2-4)}{(x^2+4)^2}$$

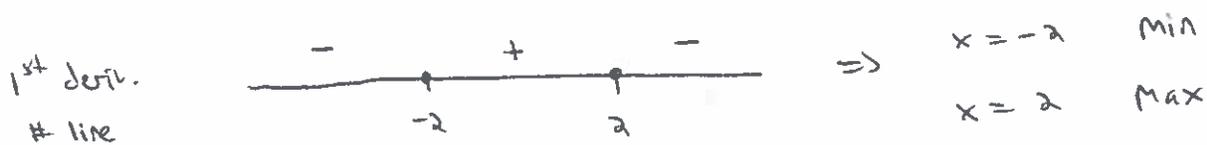
$$f''(x) = \frac{(x^2+4)^2 \cdot -6x - (-3x^2+12) \cdot 2(x^2+4) \cdot 2x}{(x^2+4)^4}$$

$$= \frac{-6x(x^2+4)^2 - 4x(-3x^2+12)(x^2+4)}{(x^2+4)^3}$$

$$= \frac{-6x^3 - 24x + 12x^3 - 48x}{(x^2+4)^3}$$

$$= \frac{6x^3 - 72x}{(x^2+4)^3} = \frac{6x(x^2-12)}{(x^2+4)^3}$$

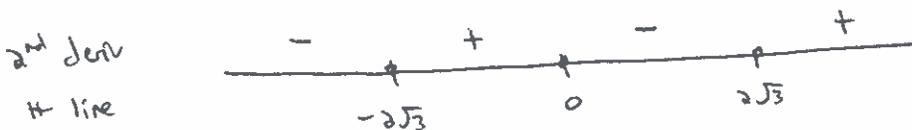
$$c) f' = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$$



$$(-2, -\frac{3}{4}) \text{ Min}, \quad (2, \frac{3}{4}) \text{ Max}$$

$$d) f'' = 0 \Rightarrow 6x(x^2 - 12) = 0 \Rightarrow 6x(x - \sqrt{12})(x + \sqrt{12}) = 0$$

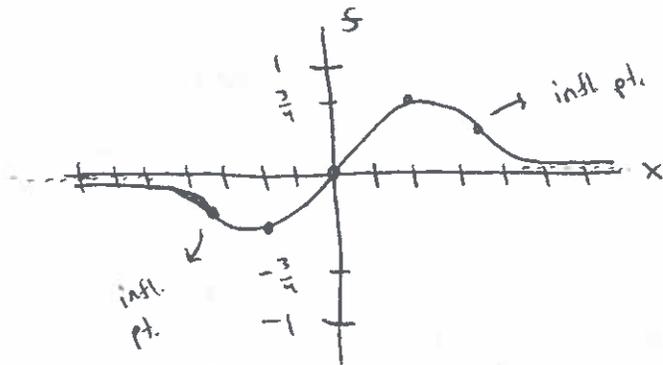
$$\Rightarrow x = 0, \pm \sqrt{12} \text{ or } x = 0, \pm 2\sqrt{3}$$



$$\Rightarrow x = 0, \pm 2\sqrt{3} \text{ are inflection points}$$

e) Note that  $f(-x) = \frac{-3x}{(-x)^2 + 4} = -\frac{3x}{x^2 + 4} = -f(x)$  so that  $f$  is an odd function.

(symmetric about origin)



(4) a) Int. By Parts Twice:  $u = x^2$   $du = 2x dx$   $dv = \cos x dx$   $v = \sin x$

$$\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx$$

$$u = 2x \quad du = 2 dx$$

$$v = -\cos x$$

$$= x^2 \sin x + \left( +2x \cos x + \int +2 \cos x dx \right)$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

b) u-sub

$$u = t^2 + 4t + 8$$

$$du = 2t + 4 = 2(t+2)$$

$$\frac{1}{2} \int \frac{2(t+2)}{t^2+4t+8} dt = \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|t^2+4t+8| + C$$

c) Partial Fractions

$$\frac{x+13}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \Rightarrow x+13 = A(x+1) + B(x-3)$$

$$x=3 \Rightarrow 16 = 4A \Rightarrow A=4$$

$$x=-1 \Rightarrow 12 = -4B \Rightarrow B=-3$$

$$\int \frac{x+13}{(x-3)(x+1)} dx = \int \frac{4}{x-3} - \frac{3}{x+1} dx = 4 \ln|x-3| - 3 \ln|x+1| + C$$

d) Trig. Sub

$$x = \tan \theta$$

$$dx = \sec^2 \theta d\theta$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\int \frac{1}{x^2 \sqrt{x^2+1}} dx = \int \frac{1}{\tan^2 \theta \cdot \sqrt{\tan^2 \theta + 1}} \cdot \sec^2 \theta d\theta$$

$$= \int \frac{\sec \theta}{\tan^3 \theta} d\theta = \int \frac{1}{\cancel{\cos \theta}} \cdot \frac{\cos^3 \theta}{\sin^3 \theta} d\theta = \int \frac{\cos \theta}{\sin^3 \theta} d\theta$$

u-sub

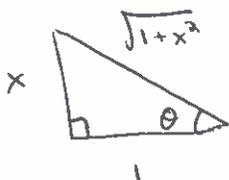
$$u = \sin \theta$$

$$du = \cos \theta d\theta$$

$$= \int \frac{1}{u^3} du = -\frac{1}{u} + C = -\frac{1}{\sin \theta} + C$$

$$\tan \theta = \frac{x}{1} \Rightarrow \sin \theta = \frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow \int \frac{1}{x^2 \sqrt{1+x^2}} dx = -\frac{\sqrt{1+x^2}}{x} + C$$



⑤ a) use  $|\sin n| \leq 1$

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^4} \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \rightarrow \text{converges since it's a p-series with } p=4$$

By the comparison test,  $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^4} \right|$  converges.

By the Absolute convergence test, the original series  $\sum_{n=1}^{\infty} \frac{\sin n}{n^4}$  converges.

b) Converges by the ratio test.

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{2} \cdot \left( \frac{n+1}{n} \right)^2 \right| = \frac{1}{2} < 1$$

c) Diverges by the  $n^{\text{th}}$ -term test.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-1)^n \frac{n+1}{n-1} \neq 0 \quad (\text{oscillates between 1 and -1})$$

⑥  $\frac{dy}{dx} = (2-y)x^2$ ,  $y(0) = 1$ ,  $\Delta x = 0.25$   
Find  $y(1)$ . (4 steps)

n	x	y	M
0	0	1	0
1	0.25	1	0.0625
2	0.5	1.0156	0.2461
3	0.75	1.0771	0.5191
4	1.0	1.2069	

$$y_1 = y_0 + M \Delta x = 1 + 0 \cdot 0.25 = 1$$

$$y_2 = y_1 + M \Delta x = 1 + 0.0625 \cdot 0.25 = 1.0156$$

$$y_3 = y_2 + M \Delta x = 1.0156 + 0.2461 \cdot 0.25 = 1.0771$$

$$y_4 = y_3 + M \Delta x = 1.0771 + 0.5191 \cdot 0.25 = 1.2069$$

$$y(1) \approx 1.2069$$

$$b) \int \frac{dy}{2-y} = \int x^2 dx \Rightarrow -\ln|2-y| = \frac{x^3}{3} + C$$

$$\Rightarrow \ln|2-y| = -\frac{x^3}{3} + C$$

$$\Rightarrow |2-y| = C e^{-x^3/3}$$

$$\Rightarrow 2-y = C e^{-x^3/3}$$

$$\Rightarrow y(x) = 2 - C e^{-x^3/3}$$

$$b) \text{ cont. } y(0) = 1 \Rightarrow 1 = a - c \Rightarrow c = 1$$

$\therefore y(x) = a - e^{-x/3}$  is the solution to the ODE with  $y(0) = 1$ .

$$c) y(1) = a - e^{-1/3} \approx 1.283469$$

Error:  $1.283469 - 1.2069 = 0.0766$  below actual value

⑦ Let  $y(t)$  = temperature of coffee after  $t$  minutes

$A$  = ambient temperature =  $20^\circ\text{C}$

$k$  = proportionality constant

$$y(0) = 99^\circ\text{C}$$

$$y(2) = 90^\circ\text{C}$$

$$\begin{aligned} \frac{dy}{dt} &= k(y - 20) \Rightarrow \int \frac{dy}{y-20} = \int k dt \Rightarrow \ln|y-20| = kt + c \\ &\Rightarrow |y-20| = ce^{kt} \\ &\Rightarrow y(t) = 20 + ce^{kt} \end{aligned}$$

$$y(0) = 99 \Rightarrow 99 = 20 + ce^{k \cdot 0} \Rightarrow c = 79$$

$$y(2) = 90 \Rightarrow 90 = 20 + 79e^{2k} \Rightarrow \frac{70}{79} = e^{2k}$$

$$\Rightarrow \ln\left(\frac{70}{79}\right) = 2k \Rightarrow k = \frac{1}{2} \ln\left(\frac{70}{79}\right) \approx -0.060476$$

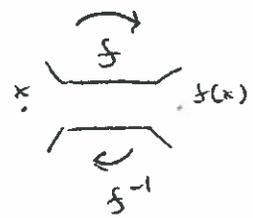
$$\therefore y(t) = 20 + 79e^{-0.060476t}$$

$$y(5) = 20 + 79e^{-0.060476 \cdot 5} = 78.39^\circ\text{C}$$

$$60 = 20 + 79e^{-0.060476t} \Rightarrow \frac{40}{79} = e^{-kt}$$

$$\Rightarrow t = \frac{1}{k} \cdot \ln\left(\frac{40}{79}\right) = \frac{2 \ln\left(\frac{40}{79}\right)}{\ln\left(\frac{70}{79}\right)} \approx 11.25 \text{ minutes}$$

8) a) TRUE Recall that  $f(f^{-1}(x)) = x$



Differentiate both sides w.r.t.  $x \Rightarrow$

$$f'(f^{-1}(x)) \cdot f^{-1}'(x) = 1 \Rightarrow f^{-1}'(x) = \frac{1}{f'(f^{-1}(x))}$$

Since  $f$  is decreasing,  $f' < 0$  always.

$$f'(f^{-1}(x)) < 0 \Rightarrow f^{-1}'(x) < 0$$

$\Rightarrow f^{-1}$  is strictly decreasing.

FALSE

b)  $\lim_{t \rightarrow 0} \frac{\cos(t^2) - 1}{t^4}$  is in the form " $\frac{0}{0}$ ". Applying L'Hôpital's Rule

gives  $\lim_{t \rightarrow 0} \frac{-\sin(t^2) \cdot 2t}{4t^3} = \lim_{t \rightarrow 0} \frac{-\sin(t^2)}{2t^2}$ , which is also " $\frac{0}{0}$ ".

Applying L'Hôpital's Rule again gives  $\lim_{t \rightarrow 0} \frac{-\cos(t^2) \cdot 2t}{4t}$

$$= \lim_{t \rightarrow 0} \frac{-\cos(t^2)}{2} = \left(-\frac{1}{2}\right) \quad \text{The limit exists and is } -\frac{1}{2}.$$

Note the importance of simplifying the fractions before applying L'Hôpital.

c) False

Let  $a_n = \frac{1}{n}$  and  $b_n = \frac{1}{n}$ .

$\sum_{n=1}^{\infty} \frac{1}{n}$ , the harmonic series, diverges but

$$\sum_{n=1}^{\infty} a_n \cdot b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \text{ converges (p-series with } p=2)$$

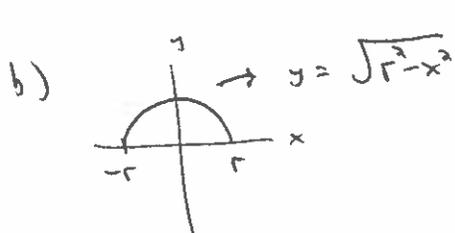
9) a)  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \rightarrow 0} \left( \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \right) \cdot \frac{(\sqrt{x+h+1} + \sqrt{x+1})}{(\sqrt{x+h+1} + \sqrt{x+1})}$$

mult. top + bottom  
by conjugate

$$= \lim_{h \rightarrow 0} \frac{x+h+1 - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \frac{1}{2\sqrt{x+1}}$$



$$V = \int_{-r}^r \pi (\sqrt{r^2 - x^2})^2 dx$$

$$= 2\pi \int_0^r (r^2 - x^2) dx$$

$$= 2\pi \left( r^2 x - \frac{x^3}{3} \Big|_0^r \right)$$

$$= 2\pi \left( \frac{2}{3} r^3 \right) = \frac{4}{3} \pi r^3 \quad \checkmark$$

c) See Next Page.

d) Plug in  $y = e^{kx}$  into the ODE.

$$y' = k e^{kx}, \quad y'' = k^2 e^{kx}$$

$$\Rightarrow k^2 e^{kx} - 3k e^{kx} - 10 e^{kx} = 0$$

$$\Rightarrow e^{kx} (k^2 - 3k - 10) = 0$$

$$\Rightarrow k^2 - 3k - 10 = 0$$

$$(k-5)(k+2) = 0$$

$$\therefore k = 5 \text{ or } k = -2$$

(9) c)  $\sum_{n=2}^{\infty} \frac{4}{n^2-1}$  is a telescoping series.

Use Partial Fractions:  $\frac{4}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1}$

$$\Rightarrow 4 = A(n+1) + B(n-1)$$

$$n = -1 \Rightarrow 4 = -2B \Rightarrow B = -2$$

$$n = 1 \Rightarrow 4 = 2A \Rightarrow A = 2$$

$$\therefore \sum_{n=2}^{\infty} \frac{4}{n^2-1} = \sum_{n=2}^{\infty} \frac{2}{n-1} + \frac{-2}{n+1}$$

$$= 2 \sum_{n=2}^{\infty} \frac{1}{n-1} + \frac{-1}{n+1}$$

$$= 2 \left[ \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots \right]$$

we have  $S_n = 2 \left[ 1 + \frac{1}{2} \right] - \frac{1}{n+1}$  so  $\lim_{n \rightarrow \infty} S_n = 2 \cdot \frac{3}{2} - 0$

$$\lim_{n \rightarrow \infty} S_n = S \Rightarrow$$

$$= \textcircled{3}$$

$$\sum a_n = S.$$