

MATH 136-04, Fall 2010

Section 5.2: The Definite Integral

Definition 0.1 If f is a continuous function on $a \leq x \leq b$, and $\Delta x = \frac{b-a}{n}$, where n corresponds to the number of subdivisions of $[a, b]$, then the **definite integral of f from a to b** is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} (f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x)$$

where x_i^* is any sample point in the sub-interval $[x_{i-1}, x_i]$.

The summation in the parentheses above is called a **Riemann Sum**. If, for each i , x_i^* is the left endpoint of the i th sub-interval, then the resulting sum is a **Left-Hand Sum**, abbreviated L_n . Alternatively, if for each i , x_i^* is the right endpoint of the i th sub-interval, then the resulting sum is a **Right-Hand Sum**, abbreviated R_n . Finally, if for each i , x_i^* is the midpoint of the i th sub-interval, then the resulting sum is a **Midpoint Sum** (called the Midpoint Rule in the text), abbreviated M_n .

As n gets larger, the width of the approximating rectangles goes to zero; however, the number of such rectangles is going to infinity. If f is continuous, this limit always exists as a finite number and represents the **signed area** under the curve. This means, for example, that if the graph of f lies entirely below the axis, then the definite integral will be negative. Thus, “negative area” is allowed here.

Some Important Notes:

- The definite integral is a **number**!
- Treating $\int_a^b f(x) dx$ as signed area, the area of the parts of f above the axis count as positive area while the area of the parts of f below the axis count as negative area. The total signed area is the sum of these two numbers.
- The variable inside the integral is irrelevant, and is usually called a **dummy variable**. Thus,

$$\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(\theta) d\theta = \int_a^b f(H_C) dH_C.$$

What matters is f and the limits of integration, not the particular variable. It is typical to think of dx as indicating to integrate with respect to x .

Example 0.2 Explain why $\int_0^{2\pi} \sin \theta d\theta = 0$.

Example 0.3 Given that the area under the sine curve between 0 and π is 2, use symmetry to find $\int_0^{11\pi} \sin \theta d\theta$ and $\int_\pi^{12\pi} \sin \theta d\theta$.

Since we already know how to compute the areas of certain familiar geometric shapes (e.g., triangles, rectangles, circles), we can evaluate certain definite integrals by graphing the integrand and calculating the area under the curve using a well-known area formula.

Example 0.4 Calculate $\int_{-2}^1 2x + 1 \, dx$ by interpreting the definite integral in terms of areas.

Example 0.5 Calculate $\int_{-3}^3 \sqrt{9 - x^2} \, dx$ by interpreting the definite integral in terms of areas.

Properties of Integrals

1. $\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx$ (interchanging the limits of integration pulls out a minus sign)

2. $\int_a^a f(x) \, dx = 0$

3. $\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx$ (linearity of integration limits)

4. $\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$ (linearity of the integral)

5. $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$ (constants pull out)

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x) \, dx \geq 0$.

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx$. (comparing integrals)

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$.

Example 0.6 Derive Property 2 from Property 1.

Example 0.7 Interpreting the integral as area, what is $\int_a^b m \, dx$? (Here, m is a fixed constant.) Use your result to derive Property 8 from Property 7.

Example 0.8 Evaluate $\int_{-2}^5 3|x - 1| \, dx$.

Example 0.9 Evaluate $\int_{-2}^0 7 - 4\sqrt{4 - x^2} \, dx$.

Example 0.10 Evaluate $\int_{-6}^6 \sin x + x^{101} - 9x^3 \, dx$. Hint: What type of function is the integrand?

Example 0.11 If $\int_1^8 f(x) \, dx = 7$ and $\int_4^8 f(x) \, dx = 10$, find $\int_1^4 5f(x) \, dx$.