MATH 136-04, Fall 2006

Computer Lab #2

Newton's Method

DUE DATE: Monday, October 23rd, Start of class

The goal for this lab project is to use and investigate the numerical root-finding algorithm known as Newton's method (sometimes referred to as the Newton-Raphson method.) Using the computer software package Maple, you will apply Newton's method to various functions in order to find their roots. In addition to computing approximate solutions to complicated equations, Maple will also give a graphical depiction of Newton's method helpful for visualizing the technique. In the process of investigating different examples, you will determine the accuracy of Newton's method as well as explore some cases where the method fails quite badly. For fun, you will use Newton's method to compute the value of π to 40 digits.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced. Please turn in **one report per group**, listing the names of the groups members at the top of your report. Be sure to answer all questions carefully and neatly, writing in complete sentences. Your report should be TYPED and you are encouraged to type all of it in your Maple worksheet. However, should you need to write in some mathematical calculations or figures, feel free to leave space in your worksheet to write them in by hand later.

Newton's Method

Newton's method makes use of the derivative to approximate a solution to the equation $f(x) = 0$. The algorithm is a very simple one, but works remarkably well. Begin with an initial guess x_0 to the equation. Find the tangent line to f at $x = x_0$ and compute where this line intersects the x-axis. Call this intersection point x_1. This is the next guess. Compute the tangent line to f at $x = x_1$ and find the place where it intersects the x-axis. Call this point x_2. Repeating the process produces a sequence of points

$$x_0, x_1, x_2, x_3, \ldots, x_n, \ldots$$

that should get closer and closer to a solution of the equation $f(x) = 0$. We say that the sequence of points $x_0, x_1, \ldots$ **converges** to a root r of the equation if

$$\lim_{n \to \infty} x_n = r.$$

We will discuss convergence later on in the course when we study infinite series.

Ideally, the iterative process described above will eventually head towards a root or zero of the equation $f(x) = 0$. Different initial seeds may have different sequences of values, but if we choose well, Newton's method usually succeeds in finding a solution. However, as you will see below, there are plenty of cases where Newton's method fails quite badly.

As discussed in class, to solve the equation $f(x) = 0$, Newton's method reduces to iterating the following expression:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

In other words, to obtain the next guess x_{n+1} in the process, we plug the previous guess x_n into the right-hand side of equation (1). Note that this computation is fairly simple, involving only the function value at x_n and the derivative at x_n. To produce the sequence of approximations to the solution $x_0, x_1, x_2, \ldots$ we repeatedly plug into the right-hand side of equation (1). This process is called **iteration**, something computers are ideally suited for.

A Simple Example

Let's begin with a simple example, finding the roots of $f(x) = x^2 - 1$. Of course we already know the answer without using Newton's method but this is a good warmup exercise. Start with an initial guess of $x_0 = 2$. Since the derivative of f is just $f'(x) = 2x$, we see that the next number in our sequence of approximations is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{3}{4} = 1.25.$$

From here we can compute x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.25 - \frac{0.5625}{2.5} = 1.025.$$

Be sure to check these calculations so you understand how the method works.

To use Maple to apply Newton's method, first load the `Student[Calculus1]` package by typing

```
with(Student[Calculus1]);
```

NOTE: Every time you start up a new session of Maple, you will need to execute this command in order to use the `NewtonsMethod` command. To apply Newton's method to our example above, type the following:

```
NewtonsMethod(x^2-1, x = 2, iterations = 2);
```

This should output the value of x_2 we found above: 1.025. The syntax for the `NewtonsMethod` command is as follows: The function $f(x)$ you want to find the roots of goes in the first entry after the parentheses (in this case $f(x) = x^2 - 1$.) The second entry is the initial seed (in this case $x_0 = 2$.) The third and remaining entries contain various options. In this case, `iterations = 2` computes two iterations of Newton's method and outputs the value x_2. Try changing this to `iterations = 1`. What does Maple return in this case? Is it correct? You can obtain more iterations of Newton's method and thus presumably a better approximation to the root by increasing the number of iterations.

To see a list of successive iterations, use the option `output = sequence`. Typing

```
NewtonsMethod(x^2-1, x = 2, iterations = 5, output = sequence);
```

for example, lists the first 5 iterations (actually 6 numbers starting with $x_0 = 2$) of Newton's method applied to $f(x) = x^2 - 1$. What do you notice about the final number in the Maple output?

To visualize successive iterations of Newton's method, use the option `output = plot`. Typing

```
NewtonsMethod(x^2-1, x = 2, iterations = 5, output = plot);
```

for example, produces a graph of the function f along with tangent lines showing successive points (in green) converging towards the root $x = 1$.

Finally, it is possible to view each iteration at a time, using the option `output = animation`. Typing

```
NewtonsMethod(x^2-1, x = 2, iterations = 5, output = animation);
```

for example, produces a graph of the parabola f along with a dashed vertical line going from $x = 2$ to the graph. Clicking on the graph produces DVD style control buttons on the upper-left hand corner of your Maple window. By clicking on the next frame button `>|` (this is the button directly to the left of the words "Current Frame") you plot the next iteration of Newton's method.

Lab Exercises

1. Continuing with the example above, $x^2 - 1 = 0$, what happens if you apply Newton's method starting with an initial guess of $x_0 = -2$? In other words, what number does the sequence of iterates approach? What happens with an initial guess of $x_0 = 0$? (Try using the `output = plot` option here.) Explain why this happens. By looking at a graph of $f(x) = x^2 - 1$, determine which initial seeds converge to the root at $x = 1$ and which initial seeds converge to the root at $x = -1$. Confirm your answer by testing different seeds with the `NewtonsMethod` command.

2. **Effectiveness of Newton's Method:** Let's explore how well Newton's method converges to a root by computing the error at each step of the process. First execute the command `Digits := 20` which tells Maple to compute all numerical values out to 20 decimal places. Next, use Maple to compute the first six iterations of Newton's method applied to $f(x) = x^2 - 1$ with initial seed $x = 2$. What is the error at each step, that is, how far is each iteration from the true solution $x = 1$? Give your answers in standard form, eg. 2.3×10^{-3}. How much does Newton's method improve at each step? How many decimal places of accuracy are gained at each step?

3. **Computing π:** Use Newton's method to compute the first 40 digits of π. Give both the answer and the Maple command you used to find it. *Hint:* $r = \pi$ is the root of what well-known function? Be sure to set the value of `Digits` high enough.

4. Use Newton's method to compute the first 20 digits of the fifth root of 2. Give both the answer and the Maple command you used to find it.

5. Find all real solutions (to 10 decimal places) of the equation

$$3\cos(x^2 + 2) = x.$$

   Hint: Try plotting a graph of f to find good initial guesses. See the *Introduction to Maple Computer Labs* handout for help with plotting functions.

6. **Complete Success:** Explain why Newton's method finds the solution to a linear equation in exactly one iteration, no matter what the initial seed is. In other words, if $f(x) = mx + b$ (with $m \neq 0$), then for any choice of x_0, Newton's method finds the only root of $f(x) = 0$ in precisely one iteration.

7. **Partial Failure of Newton's Method:** What happens when you apply Newton's method to find the roots of $p(x) = x^4 - 6x^2 - 11$ with initial seed $x = 1$? Print out a plot using the `NewtonsMethod` command with the `output = plot` option, illustrating what goes wrong in this case. Suppose you try to avoid this problem by changing the initial seed slightly to $x_0 = 0.8$ or to $x_0 = 1.25$. Does this help? By making better initial guesses, find the real roots (to 10 decimal places) of $p(x)$.

8. **Complete Failure:** Explain why Newton's method fails to find any roots of $f(x) = x^{1/3}$ when starting with any initial seed $x_0 \neq 0$.