MATH 136-01, Fall 2004

Group Lab Project #4

Machin's Formula and the Value of π

DUE DATE: Friday, Dec. 3rd, in class.

The goal for this lab project is to develop a better understanding of Taylor series and their convergence by exploring two famous formulas for π. It is **required** that you work in a group of two people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. Please turn in one report per group, listing the names of the groups members at the top of your report. Be sure to answer all questions carefully and neatly, writing in complete sentences. You should turn in your answers on separate sheets of paper.

The Beauty of π

There is no number that has fascinated more people, appeared in more places and been the subject of more research than the number π. The ratio between the circumference and the diameter of a circle has had obvious physical significance throughout human history. It also appears in places you might not expect. The number π is indirectly referred to in the bible. It appeared in the O. J. Simpson trial in 1995 to discredit an FBI special agent who miscalculated the area of a circle. Many Macintosh computer programmers write their source code with files ending in π such as "mycode.π" There have even been laws nearly passed about π (Indiana state legislature, 1897) and silly jokes such as "What do you get if you divide the circumference of a jack-o-lantern by its diameter? Answer: pumpkin pie."

The number π is irrational, that is, its decimal expansion never stops and never repeats. It is also transcendental, meaning that it cannot be written as the solution to a polynomial equation with integer coefficients. For example, $\sqrt{2}$ is not transcendental because it is a solution to $x^2 = 2$. Interestingly, Euler's famous formula

$$e^{i\pi} + 1 \;=\; 0$$

is used to prove that π is irrational and transcendental. Because of the importance of π and its connection to the circle (a spiritual symbol for many), accurately calculating as many digits of π as possible has become a rite of passage for mathematicians past, present and future.

The first 50 decimal places of π are

$$\pi \;=\; 3.14159265358979323846264338327950288419716939937510$$

There is a world record for computing the most number of digits of π. In September of 2002, a team of researchers led by Dr. Kanada at the University of Tokyo computed π to 1.2411 trillion digits. This appears to be the current world record. There are literally hundreds of formulas for computing π and the challenge is to find the one or two that converge the fastest and are the easiest to use. In this project you will contrast two well-known formulas due to Leibniz and Machin.

The Project

Here are two famous formula's for π:

$$\frac{\pi}{4} \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - + \cdots$$

discovered by Leibniz in 1674 and

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$$

discovered by the British mathematician John Machin in 1706. Of course, to find π we multiply each formula by 4. We derived the Leibniz formula in class using the Taylor series for $f(x) = \arctan x$. It is therefore not entirely surprising to see the use of arctan in Machin's formula. In this lab you will derive Machin's formula.

The Leibniz formula exquisitely relates the odd natural numbers to π. However, it is an infinite series whereas the formula by Machin involves only two terms. On the other hand, to approximate π using Machin's formula, we need to approximate the arctan function, and this in turn, requires an infinite series. So both formulas really involve an infinite series. The question is, which series does a better job of computing the digits of π?

Historical note: In 1873, William Shanks calculated π to 707 digits using a formula similar to Machin's. However, he made a mistake in the 527th digit and all the remaining digits were wrong! The error was not found until 1946 so π was listed incorrectly for 73 years. This and many other interesting facts about π can be found in the delightful book *The Joy of* π by David Blatner. There is also a companion website at `http://www.joyofpi.com`.

1. Begin by computing the sum of the first 10 terms in Leibniz's formula to obtain an approximation for π. How good is this approximation? In other words, how many digits of π are given correctly? (Start counting digits from the decimal place.) How much does the approximation improve if you include another 10 terms? Based on your findings, is Leibniz's formula quickly or slowly approaching the actual value for π?

2. Recall the Taylor series expansion for $f(x) = \arctan x$,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - + \cdots$$

   (a) Use the first five terms of this series to approximate $\arctan(\frac{1}{5})$ and $\arctan(\frac{1}{239})$. Use your two estimates and Machin's formula to approximate π. How good is this approximation? How many digits of accuracy are there?

   (b) Now use the first ten terms of the Taylor series for $\arctan x$ to approximate $\arctan(\frac{1}{5})$ and $\arctan(\frac{1}{239})$. Use your two estimates and Machin's formula to approximate π. How good is this approximation? How many digits of accuracy are there now?

   Note: You may or may not be able to do this computation on your calculator depending upon its degree of accuracy. You can use MAPLE quite easily however. First define a polynomial $p(x)$ using `p := x -> x - x^3/3 + x^5/5 - + ...` where p can be the first 5 or 10 terms in the Taylor series of $\arctan x$. Then `p(1/5)` gives the function value at 1/5 as a fraction. To get a numerical answer use `evalf(p(1/5), 20)` which will give you the value to 20 decimal places. For example, `evalf(Pi,100)` gives the first 100 digits of π. Then, you can use Machin's formula with $p(x)$ instead of $\arctan x$ to get an estimate for π with as many decimal places as you like.

3. Based on your findings in the previous two questions, which formula, Leibniz's or Machin's, gives a better, quicker way of calculating π. Using your knowledge of Taylor series, why does one formula converge faster than the other?

4. Some properties of $\tan x$:

(a) Is $f(x) = \tan x$ an even or odd function? or perhaps neither? Explain with a short calculation.

(b) Use the angle addition formulas for $\cos x$ and $\sin x$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \qquad (1)$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad (2)$$

to derive the angle addition formula for $\tan x$:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad (3)$$

5. Deriving Machin's formula:

(a) Using formula (3) with $A = \arctan(120/119)$ and $B = -\arctan(1/239)$, show that

$$\arctan(\frac{120}{119}) - \arctan(\frac{1}{239}) = \arctan(1)\,.$$

(b) Using formula (3) with $A = B = \arctan(1/5)$, show that

$$2\arctan(\frac{1}{5}) = \arctan(\frac{5}{12})\,.$$

Use a similar method to show that

$$4\arctan(\frac{1}{5}) = \arctan(\frac{120}{119})\,.$$

(c) Derive Machin's formula for π.

References

1. Blatner, David, *The Joy of* π, Walker Publishing Company, Inc. 1997

2. "The Joy of π" website, http://www.joyofpi.com