MATH 136-01
Chapter 9 Topic Review Sheet

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This is a list of terminology and topics covered in the ninth chapter of *Calculus*, D. Hughes-Hallet, et al. 3rd edition. Please consult the text for full definitions, statements of properties, and numerous examples and exercises. Terms in bold face are defined in the text. The primary focus of Chapter 9 is understanding infinite series and determining whether they converge or diverge.

Geometric Series (Section 9.1) A **geometric series** is a sum where each term is obtained by multiplying the previous term by a fixed ratio r. The sum of a finite geometric series is given by

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

provided that $r \neq 1$. An infinite geometric series **converges** if and only if $|r| < 1$. The sum of an infinite geometric series is given by

$$S = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \frac{a}{1-r}$$

provided that $|r| < 1$. This formula is easily obtained by taking the limit as $n \to \infty$ of the previous formula.

Convergence of Sequences and Series (Section 9.2) A **sequence** is a list of numbers, such as, $S_1, S_2, S_3, \ldots$. A **series** is a sum of numbers. These are very different things! We need to understand how to compute the limit of sequences in order to determine whether an infinite series $\sum_{n=1}^{\infty} a_n$ converges or diverges. The **partial sum** of a series is the sum of the first n terms of a series

$$S_n = a_1 + a_2 + \cdots + a_n.$$

The larger n is, the more terms in the series are being added together. To determine whether an infinite series **converges** or **diverges** we take the limit of the partial sums. Specifically, if $\lim_{n\to\infty} S_n = S$ exists and is a finite number, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges and its sum is S. If $\lim_{n\to\infty} S_n$ does not exist or is infinite, we say that the series diverges.

Perhaps the most important first test for determining whether a series converges is the **nth term test**. Given a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. This test is really a test for divergence. If the terms being added in the series do not approach zero, then the series cannot possibly converge.

IMPORTANT: If $\lim_{n\to\infty} a_n = 0$ we know nothing about the convergence of the series! The series could converge or diverge. We must use other tests to determine which case it is.

The **Integral Test** can be used to determine whether a series converges or diverges. If $c \geq 0$ and $f(x)$ is a decreasing positive function defined for all $x \geq c$ with $a_n = f(n)$ for all n, then:

- If $\int_c^{\infty} f(x)\,dx$ converges, then $\sum a_n$ converges.
- If $\int_c^{\infty} f(x)\,dx$ diverges, then $\sum a_n$ diverges.

One very important series which diverges is the **harmonic series**:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

This series can be shown to diverge using the integral test. It is an important series because the terms $a_n = 1/n$ go to zero as $n \to \infty$ but the partial sums still approach ∞.

Tests for Convergence (Section 9.3)

The **Comparison Test** states that if $0 \leq a_n \leq b_n$ for all n, then:

- If $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges.

The **Ratio Test** is derived by comparing the "tail" of an infinite series with a geometric series. Since the tail of an infinite series controls convergence, this is often a useful test. For a series $\sum a_n$, suppose that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \;=\; L$$

exists or is infinite. This means that the series starts looking geometric with ratio L. Then:

- If $L < 1$, then $\sum a_n$ converges.
- If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.
- If $L = 1$, the ratio test is inconclusive.

An **alternating series** is a series for which the terms alternate in sign. These series are often written in the form $\sum(-1)^{n-1}a_n$ where $a_n > 0$. The **Alternating Series Test** states that if a_n is a positive, decreasing sequence approaching 0 as $n \to \infty$, then the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

converges.

Power Series (Section 9.4) A **power series** is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n \;=\; c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

The series is being expanded around the constant a with x treated as a variable. Thus, a power series is really a family of infinite series, one for each value of x. Power series have a **radius of convergence** R which determines how far from a on the number line the series converges. If $R = 0$, the power series converges only for $x = a$. If $R = \infty$, the power series converges for all x. Otherwise, the series converges for all x satisfying $|x - a| < R$. This inequality reads "the distance between x and a is less than R" and is therefore equivalent to $x \in (a - R, a + R)$. The series may or may not converge at the endpoints $x = a - R$ and $x = a + R$.

To determine the radius of convergence R for a power series, use the ratio test. Letting $a_n = c_n(x-a)^n$, compute

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \;=\; L.$$

If $L = \infty$, then $R = 0$. If $L = 0$, then $R = \infty$. Otherwise, L will be of the form $k|x - a|$ for some positive constant k. In this case, solving $L < 1$ yields $|x - a| < 1/k$ and thus $R = 1/k$.