MATH 136-01
Chapter 8 Topic Review Sheet

November 23, 2004

This is a list of terminology and topics covered in the eighth chapter of *Calculus*, D. Hughes-Hallet, *et. al.* 3rd edition. Please consult the text for definitions, statements of properties, and numerous examples and exercises. Terms in bold face are defined in the text. We covered sections 8.1, 8.2 and 8.3.

Areas and Volumes. (Section 8.1) In this section, we calculate volumes of solid objects using definite integrals. We first approximate the volume by slicing up the region and construct a Riemann sum which adds up the volumes of the individual slices. Then, we take the limit as the number of slices approaches infinity (i.e. the thickness of each slice approaches zero). This results in a definite integral which gives the exact volume.

More precisely, we must slice the object in a way that allows us to easily determine the volume of a slice. If Δx represents the thickness of a slice, we want the cross-sectional area $A(x)$ of a slice at position x to have a simple formula. This will be the case if the cross-section of our slice is a geometric shape such as a circle, square, or triangle. In a Riemann sum, we have

$$\text{Volume} \approx \sum_{i=0}^{n} A(x_i)\,\Delta x$$

and letting $n \to \infty$, we have

$$\text{Volume} = \int_a^b A(x)\,dx \tag{1}$$

where $x = a$ is the starting point of the first slice, and $x = b$ is the ending point of the last slice. If you like, you can think of this in terms of the following variable-free formula:

$$\text{Volume} = \int_{\text{start of stack}}^{\text{end of stack}} \begin{pmatrix}\text{cross-sectional} \\ \text{area of slice}\end{pmatrix} \cdot \text{thickness}. \tag{2}$$

Volumes of Solids of Revolution, and Arclength. (Section 8.2) For the first application in this section, we compute volumes of **solids of revolution**, using definite integrals as we did in Section 8.1. In this case, a region R in the xy-plane is revolved about an axis, giving a solid whose slices will be one of the following:

- *Discs:* If the region R maintains contact with the axis of revolution throughout, then the cross-section of each slice will be a solid disc whose radius will be given in terms of one of the curves bounding R.
- *Washers:* If the region R does not touch the axis of revolution throughout, then the cross-section of each slice will be a *washer* whose inner and outer radii will be given in terms of the curves bounding R.

In either case, we may use Equation 2 above, once we determine the relevant pieces to put into the integral. To achieve this, we do the following:

1. Sketch the region R, note the axis of revolution, and sketch in a "generating strip" (i.e. a thin strip which will produce a slice when revolved about the axis of revolution).

2. Determine the variable of integration. If we are revolving about the x-axis, or a line parallel to the x-axis, then the variable of integration will be x, in order for the cross-sections to be discs or washers. On the other hand, if we are revolving about the y-axis, or a line parallel to the y-axis, then the variable of integration will be y.

3. Find the bounds of integration, a and b. These will be found from the boundary curves of R. If they are lines, such as $x = a$ and $x = b$, then we simply use these values as our bounds. Otherwise, we may need to determine the intersection points of two boundary curves, by setting them equal to each other and solving for the variable of integration.

4. Find the cross-sectional area of a slice. First determine whether slices are discs or washers. Supposing the variable of integration is x, then the cross-sectional area of a slice at point x will be one of the following:

   - $A(x) = \pi(r(x))^2$, if slices are discs. The radius $r(x)$ is the height of the generating strip, which will be given in terms of one of the boundary curves of R.
   - $A(x) = \pi(r_{out}(x))^2 - \pi(r_{in}(x))^2$ if slices are washers. The outer radius $r_{out}(x)$ is determined by measuring the distance between the outer edge of the generating strip and the axis of revolution. The inner radius $r_{in}(x)$ is determined by measuring the distance between the inner edge of the generating strip and the axis of revolution. Both of these measurements will involve boundary curves of R in some way.

   Proceed similarly to find cross-sectional area $A(y)$ if the variable of integration is y instead of x.

5. Set up the definite integral as in Equation 2, and evaluate, using one of our methods of integration from Chapter 7. You may find it helpful to simplify the integrand first.

The second application in this section is to lengths of curves. Formulas are given for arc length of a curve, and arc length of a parameterized curve. We did not cover this material.

Density, Mass, and Center of Mass. (Section 8.3) The first application in this section involves finding a total quantity, given the density of that quantity per unit length, area, or volume. The basic model for computing total quantity from density involves a Riemann sum, and then a definite integral, similar to the volume problems in Section 8.1 and 8.2. First, we slice the given region in such a way that the density is approximately constant on each slice, and then adding up (density)·(amount of space in a slice). This gives a Riemann sum approximating the total quantity. Second, taking the limit as the number of slices approaches infinity gives the exact total quantity, represented by the following variable-free definite integral:

$$\text{Total quantity} = \int_{\text{start of slices}}^{\text{end of slices}} \begin{pmatrix} \text{density on} \\ \text{a slice} \end{pmatrix} \cdot \begin{pmatrix} \text{amount of space} \\ \text{in a slice} \end{pmatrix}. \tag{3}$$

Determining the way to slice the region is based on the density function as given. We need density to be approximately constant throughout a slice. Some common examples of regions and the way to slice them are given below:

- The region is a long thin object like a metal rod or a stretch of road, and density is given as a function $\delta(x)$ in numbers per unit length. In this case, we slice the region into small subintervals, each of length Δx. For the amount of space in a slice, we use Δx in the Riemann sum, and this becomes dx in the definite integral.

- The region is a disc, and density is given as a function $\delta(r)$ in mass per unit area, where r is the radial distance from the center of the region. In this case, density is approximately constant on thin concentric rings around the center of the region, so we slice the region into such rings, each of width Δr. For the amount of space in a slice, we approximate the area of the thin ring by stretching it out into a rectangle of length $2\pi r$ (circumference) and width Δr. The area of a slice therefore goes into the Riemann sum as $2\pi r\Delta r$, and in the definite integral it becomes $2\pi r\,dr$.
- The region is a rectangular box with specified dimensions $\ell \times w \times h$, and density is given as a function $\delta(z)$ in mass per unit volume, where z is the vertical distance from the bottom of the box. In this case, density is approximately constant on slices parallel to the bottom of the box, so we slice the region into rectangular slices, each of thickness Δz. The volume of a slice therefore goes into the Riemann sum as $\ell w \Delta z$, and in the definite integral it becomes $\ell w\, dz$.

The second application in this section involves finding the **center of mass** of either a system of point masses located at various positions along an axis, or of an object with known density function. The **center of mass**, $\overline{x}$, of a system of n discrete masses $m_1, m_2, \ldots, m_n$ lying along the x-axis at positions $x_1, x_2, \ldots, x_m$ respectively is given by:

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} x_i} = \frac{x_1 m_1 + x_2 m_2 + \cdots + x_n m_n}{m_1 + m_2 + \cdots + m_n}$$

Summing up (position)*(mass) and dividing by the total mass gives us a *weighted average* of all the masses, hence telling us the "balancing point" of the system.

The **center of mass**, $\overline{x}$, of an object lying along the x-axis between $x = a$ and $x = b$, with mass density $\delta(x)$, is given by:

$$\overline{x} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx}$$

The **center of mass**, $(\overline{x}, \overline{y})$, of a region in the plane with constant mass density δ, is given by:

$$\overline{x} = \frac{\int x\,\delta\,A_x(x)\,dx}{\text{Total Mass}} \qquad \overline{y} = \frac{\int y\,\delta\,A_y(y)\,dy}{\text{Total Mass}}$$

where $A_x(x)$ represents the length of strips perpendicular to the x-axis and $A_y(y)$ represents the length of strips perpendicular to the y-axis.