MATH 136-01

Chapter 7 Topic Review Sheet

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This is a list of terminology and topics covered in the seventh chapter of *Calculus*, D. Hughes-Hallet, *et. al.* 3rd edition. Please consult the text for definitions, statements of properties, and numerous examples and exercises. Terms in bold face are defined in the text. We will cover sections 7.1, 7.2, 7.4, 7.7 and 7.8.

Integration by Substitution (Section 7.1) Integration by substitution is the chain rule "run backwards." The chain rule tells us that $(k(j(x))' = k'(j(x))j'(x)$, so that the anti-derivative of $k'(j(x))j'(x)$ is $k(j(x))$. That is,

$$\int k'(j(x))j'(x)\ dx = k(j(x)) + C.$$

To make use of this, we must be able to recognize when an integrand is the product of two functions, one of which is a composition and the other is a constant multiple of the derivative of the inside function of the composition. In practice, if we have an integral of the form,

$$\int f(g(x))h(x)\ dx$$

we can proceed with the method if $g'(x)$ is a multiple of $h(x)$. That is, $g'(x) = c \cdot h(x)$ where c is a constant. To carry out the substitution, we let $u = g(x)$. Then $\frac{du}{dx} = c \cdot h(x)$, or $\frac{1}{c}du = h(x)dx$. Substituting both these terms into the integral we have:

$$\int f(g(x))h(x)\ dx = \frac{1}{c}\int f(u)\,du$$

If we can find an anti-derivative $F(u)$ of $f(u)$, the solution is

$$\frac{1}{c}F(u) + C = \frac{1}{c}F(g(x)) + C$$

It is important that the final answer be expressed in terms of the original variable of integration. When we apply this to a definite integral, we can either evaluate the anti-derivative at the endpoints of integration in terms of the original variable of integration, so that

$$\int_a^b f(g(x))h(x)\ dx = \frac{1}{c}F(g(x))|_a^b,$$

or convert the limits to w and then substitute into the intermediate form so that

$$\int_a^b f(g(x))h(x)\ dx = \frac{1}{c}F(u)|_{g(a)}^{g(b)},$$

Integration by Parts (Section 7.2) Integration by parts is a consequence of the product rule for derivatives. The product rule can be rearranged as follows:

$$u(x)v'(x) = (u(x)v(x))' - v(x)u'(x)$$

Taking the indefinite integral of both sides of the equation gives the following integral formula:

$$\int u(x)v'(x)\ dx = u(x)v(x) - \int v(x)u'(x)\ dx.$$

In short-hand, we have $\int u\,dv = uv - \int v\,du$. To apply this to the integral of a product, we must decide which factor we want to treat as the derivative. We then find the anti-derivative of this term, differentiate the other term, to obtain the right hand side in the above formula. The rough rule is that $u'(x)$ should be simpler than $u(x)$ and $v(x)$ should be no more complicated that $v'(x)$. If the technique is to work, the integral on the right side must be simpler (or no worse) than the one on the left side. We may have to apply this technique repeatedly in order to produce an integral that can be evaluated by other means. If, when we repeat this process, the original integral reappears on the right side, we collect the integrals on one side of the equation and divide by the coefficient to produce an answer. This is the case when we have a product of a sine or a cosine and an exponential.

Algebraic Identities and Trigonometric Substitutions (Section 7.4) The starting point in this section is the **partial fraction decomposition** of a rational function $\frac{P(x)}{Q(x)}$, when the degree of the numerator is less than that of the denominator. (If the degree of the numerator is greater than that of the denominator, we carry out polynomial long division, dividing $Q(x)$ into $P(x)$, to produce a polynomial plus a remainder term. The remainder term will be of the correct form.) Each summand of the partial fraction decomposition can be integrated by elementary techniques. The form of the decomposition is determined by the factors of $Q(x)$. The most general form of the decomposition is rather lengthy, so following the text, we write out the different pieces of the decomposition.

- If $Q(x)$ is the product of distinct linear factors, for each factor there will be a summand of the form
$$\frac{A}{x-c}.$$
- If $Q(x)$ contains a repeated linear factor $(x-c)^n$, then the decomposition must contain n summands of the form
$$\frac{A_1}{x-c} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_n}{(x-c)^n},$$
where $A_1, \ldots, A_n$ are distinct constants.
- If $Q(x)$ contains a quadratic term $q(x)$ that has no real roots, then the decomposition must contain a term of the form
$$\frac{Ax+B}{q(x)}.$$

Once we write out the correct general form for the partial fraction decomposition, we must solve for the constants in the numerator. We do this first by putting all terms over the common denominator $Q(x)$. Then we have two options, we can first collect all the terms in the numerator, set this numerator equal to $P(x)$, equate the coefficients of the terms of the same degree, and solve the corresponding system of equations. Or, we can equate numerators, evaluate the numerators at as many different values of x as we have unknown constants to produce a system of equations. In practice, choosing to substitute roots of $Q(x)$ considerably simplifies the calculations. The anti-derivative of the terms on the right will be natural logarithms, powers, or inverse tangents.

Trigonometric substitutions are also considered in this section. For integrands involving $\sqrt{a^2 - x^2}$, we replace x by $a\sin(\theta)$, so that $\sqrt{a^2 - a^2\sin^{@}(\theta)} = a\cos(\theta)$. To complete the substitution we must replace dx by $a\cos(\theta)d\theta$. For integrands involving $\frac{1}{x^2+a^2}$, we replace x by $a\tan(\theta)$, so that $\frac{1}{x^2+a^2} = \frac{1}{a^2\tan^2(\theta)+a^2} = \frac{1}{a^2\sec^2(\theta)} = \frac{1}{a^2}\cos(\theta)$. To complete the substitution we must replace dx by $a\sec^2(\theta)d\theta$.

Improper Integrals (Section 7.7) A definite integral is called an **improper integral** if either of the limits of integration is infinite or the integrand is unbounded. If a limit of integration is infinite, we replace the infinite limit by a symbolic constant, evaluate the new definite integral as we usually do, which will yield an expression in terms of this constant, then take the limit of this expression as the constant approaches $-\infty$ or ∞. For example, if the upper limit of integration is ∞, we have

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx.$$

If this limit exists, we say that the improper integral is **convergent**. Otherwise we say it is **divergent.** If the lower limit of integration is $-\infty$ we proceed in a similar manner. If both limits are infinite, then we must treat each limit separately by writing

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^0 f(x)\ dx + \int_0^\infty f(x)\ dx.$$

In the second type of improper integral, if the function is unbounded near a point, that is, has a vertical asymptote at the point, we may integrate with that point as a limit of integration. For example, if f has a vertical asymptote at 3 and we wanted to integrate f on the interval $0 \le x \le 3$, we would replace 3 by a symbolic constant and proceed as follows:

$$\int_0^3 f(x)\ dx = \lim_{b\to 3-} \int_0^b f(x)\ dx.$$

If the vertical asymptote is between the limits of integration we rewrite the integral as a sum split at the asymptote and apply this this technique to each new integral. For example, if $f(x)$ is undefined at $x = 3$, and we wanted to integrate f on the interval $0 \le x \le 5$, we would proceed as follows:

$$\int_0^5 f(x)\ dx = \lim_{b\to 3-} \int_0^b f(x)\ dx + \lim_{a\to 3+} \int_a^5 f(x)\ dx.$$

Comparison of Improper Integrals (Section 7.8) It is possible to determine whether or not an improper integral converges by comparison to known improper integrals. It is based on the following two facts:

- If $0 \le f(x) \le g(x)$ for $a \le x < \infty$ and $\int_a^\infty g(x)\ dx$ converges, then $\int_a^\infty f(x)\ dx$ converges.
- If $0 \le g(x) \le f(x)$ for $a \le x < \infty$ and $\int_a^\infty g(x)\ dx$ diverges, then $\int_a^\infty f(x)\ dx$ diverges.

There are similar comparisons for improper integrals of the second type. When using these facts, we assume that we are given an integral involving a function $f(x)$. We must then select a $g(x)$ to use for the comparison. Useful choices of $g(x)$ are power functions and exponentials.